

Laminations from the symplectic double

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Abstract

Let S be a compact oriented surface with boundary together with finitely many marked points on the boundary, and let S° be the same surface equipped with the opposite orientation. We consider the *double* $S_{\mathcal{D}}$ obtained by gluing the surfaces S and S° along corresponding boundary components. We define a notion of lamination on the double and construct coordinates on the space of all such laminations. We show that this space of laminations is a tropical version of the symplectic double introduced by Fock and Goncharov. There is a canonical pairing between our laminations and the positive real points of the symplectic double. We derive an explicit formula for this pairing using the F -polynomials of Fomin and Zelevinsky.

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1 Introduction

1.1 Previous work on laminations

As part of their investigation of higher Teichmüller theory, Fock and Goncharov studied two versions of the Teichmüller space of a surface S with boundary [FG1, FG3]. These are called the Teichmüller \mathcal{A} - and \mathcal{X} -spaces and are denoted $\mathcal{A}^+(S)$ and $\mathcal{X}^+(S)$, respectively. The Teichmüller \mathcal{A} -space, also known as the decorated Teichmüller space, was originally studied by Penner [P]. It arises as the set of positive real points of a certain algebraic variety constructed in [FG1]. The Teichmüller \mathcal{X} -space, sometimes called the enhanced Teichmüller space, is a slight variant of the classical Teichmüller space of a punctured surface. It also arises as the set of positive real points of an algebraic variety [FG1].

In addition to the Teichmüller \mathcal{A} - and \mathcal{X} -spaces, Fock and Goncharov studied two types of laminations on a surface [FG1, FG3]. The two types of laminations are called \mathcal{A} - and \mathcal{X} -laminations, and the spaces of these laminations are denoted $\mathcal{A}_L(S)$ and $\mathcal{X}_L(S)$, respectively. These laminations are closely related to Thurston's transversally measured laminations [Th]. A rational lamination is defined in [FG1, FG3] as a collection of finitely many simple nonintersecting curves on S with rational weights, subject to certain conditions and equivalence relations. The lamination spaces arise as the sets of rational tropical points of the two algebraic varieties constructed in [FG1].

In [FG1, FG3], Fock and Goncharov defined special coordinates A_i ($i \in I$) and X_i ($i \in I$) on the Teichmüller spaces $\mathcal{A}^+(S)$ and $\mathcal{X}^+(S)$, respectively. The A_i are the same as Penner's lambda lengths [P], while the X_i are related to shear coordinates. Here I is the set of edges of an ideal triangulation of the surface S , and there are certain rules for transforming between the coordinates associated with different triangulations. It was shown in [GSV, FG1] that in the case of the \mathcal{A} -space these transformation rules are examples of the cluster mutation formulas from the theory of cluster algebras pioneered by Fomin and Zelevinsky [FZI].

Similarly, there are coordinates a_i ($i \in I$) and x_i ($i \in I$) on $\mathcal{A}_L(S)$ and $\mathcal{X}_L(S)$ [FG1, FG3]. As in the case of the Teichmüller spaces, these coordinates transform according to certain rules when we change the triangulation of the surface. These rules are obtained from the cluster transformation rules by a tropicalization procedure that will be made precise later.

In addition to defining special coordinates on the Teichmüller and lamination spaces, Fock and Goncharov showed that the Teichmüller and lamination spaces are dual in a certain sense [FG1]. They defined natural subsets $\mathcal{A}_L(S, \mathbb{Z}) \subseteq \mathcal{A}_L(S)$ and $\mathcal{X}_L(S, \mathbb{Z}) \subseteq \mathcal{X}_L(S)$ and canonical maps

$$\begin{aligned} \mathbb{I} : \mathcal{A}_L(S, \mathbb{Z}) \times \mathcal{X}^+(S) &\rightarrow \mathbb{R}_{>0} \\ \mathbb{I} : \mathcal{X}_L(S, \mathbb{Z}) \times \mathcal{A}^+(S) &\rightarrow \mathbb{R}_{>0} \end{aligned}$$

which we denote by the same symbol \mathbb{I} . We usually think of these maps as operations which assign to a lamination l the function on Teichmüller space given by $\mathbb{I}(l)(m) = \mathbb{I}(l, m)$.

The functions $\mathbb{I}(l)$ obtained in this way are interesting for algebraic reasons. As shown in [FG1], the function $\mathbb{I}(l)$ can always be written as a Laurent polynomial with positive

integral coefficients in the variables $X_i^{1/2}$ or A_i . Moreover, if we consider a lamination l that has *integral coordinates*, then $\mathbb{I}(l)$ can in fact be written as a Laurent polynomial with positive integral coefficients in the coordinates X_i or A_i .

In subsequent work, Fock and Goncharov studied an algebraic variety called the symplectic double [FG4] and found that its properties are related to the geometry of a doubled surface [FG6]. To define the double $S_{\mathcal{D}}$ of a compact oriented surface S , we let S° be the same surface equipped with the opposite orientation. Then $S_{\mathcal{D}}$ is defined to be the surface obtained by gluing S and S° along corresponding boundary components. In [FG6], Fock and Goncharov considered a space $\mathcal{D}^+(S)$ which we will call the Teichmüller \mathcal{D} -space. It is similar to the Teichmüller space of $S_{\mathcal{D}}$, and it arises as the set of positive real points of the symplectic double.

To prove this fact, Fock and Goncharov showed in [FG6] that $\mathcal{D}^+(S)$ has a system of coordinates B_i, X_i ($i \in I$) associated to an ideal triangulation of S . There are certain formulas describing how the coordinates change when one changes the triangulation, and these are the same formulas used in [FG4] to define the symplectic double.

1.2 The present work

The goal of the present paper is to define and study a new class of objects called \mathcal{D} -*laminations*. A \mathcal{D} -lamination is typically represented by finitely many curves on the double $S_{\mathcal{D}}$. In the case where there are no marked points on S , one has the following definition.

Definition 1.1. A *rational \mathcal{D} -lamination* on $S_{\mathcal{D}}$ is the homotopy class of a collection of finitely many simple, nonintersecting, and noncontractible closed curves with positive rational weights and a choice of orientation for each component of ∂S which meets or is homotopic to a curve. A lamination containing homotopic curves of weights a and b is equivalent to the lamination with one curve removed and the weight $a + b$ on the other.

One of the main results of this paper is the construction of a system of coordinates b_i, x_i ($i \in I$) on the space $\mathcal{D}_L(S)$ of all rational \mathcal{D} -laminations. As in the case of \mathcal{A} - and \mathcal{X} -laminations, the coordinates transform by certain rules when we change the triangulation of the surface, and these rules can be obtained by tropicalizing the formulas that define the symplectic double.

In addition to defining special coordinates on the lamination space $\mathcal{D}_L(S)$, one of the goals of this paper is to describe a duality between \mathcal{D} -laminations and the Teichmüller \mathcal{D} -space. We describe a subset $\mathcal{D}_L(S, \mathbb{Z}) \subseteq \mathcal{D}_L(S)$ and a canonical map

$$\mathbb{I} : \mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}^+(S) \rightarrow \mathbb{R}_{>0}.$$

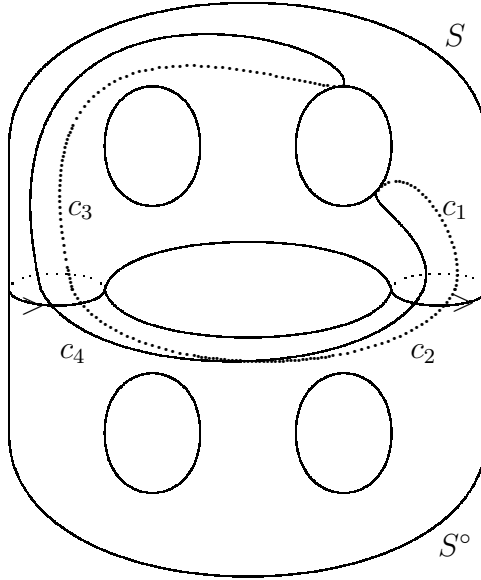
This pairing is similar to the pairings between \mathcal{A} - and \mathcal{X} -laminations and the points of the Teichmüller \mathcal{X} - and \mathcal{A} -spaces. We usually view the above pairing as a map from \mathcal{D} -laminations to functions on $\mathcal{D}^+(S)$ given by $\mathbb{I}_{\mathcal{D}}(l)(m) = \mathbb{I}(l, m)$.

We show that the function $\mathbb{I}_{\mathcal{D}}(l)$ can be expressed in a particularly nice way as a rational function in the variables B_i and $X_i^{1/2}$ for any $l \in \mathcal{D}_L(S, \mathbb{Z})$. Moreover, if we consider a

lamination l that has *integral coordinates* then $\mathbb{I}_{\mathcal{D}}(l)$ can in fact be written as a rational function in the coordinates B_i and X_i .

To make these statements more precise, let us introduce some notation and terminology. Label the elements of I by the numbers $\{1, \dots, n\}$ so that a point of $\mathcal{D}^+(S)$ has coordinates $B_1, \dots, B_n, X_1, \dots, X_n$. A curve on $S_{\mathcal{D}}$ will be called an *intersecting curve* if every curve in its homotopy class intersects the image of the boundary ∂S . Any \mathcal{D} -lamination $l \in \mathcal{D}_L(S, \mathbb{Z})$ can be represented by a collection of curves of weight 1. If this collection contains homotopic curves of weights a and b which are not intersecting curves, let us replace these by a single curve of weight $a + b$. In this way, we obtain a new collection of curves representing l . If we now cut the surface along the image of ∂S , we obtain a collection \mathcal{C} of curves on S and a collection \mathcal{C}° of curves on S° .

The illustration below shows an example of an intersecting curve on the double $S_{\mathcal{D}}$ of a genus-2 surface S with two holes. In this case, the set \mathcal{C} consists of the curves labeled c_1 and c_3 , while the set \mathcal{C}° consists of the curves labeled c_2 and c_4 . In Section 6, we will see how to associate, to any element of \mathcal{C} or \mathcal{C}° , a polynomial F_c . For curves such as the ones below, which come from an intersecting curve, these F_c are in fact the F -polynomials of Fomin and Zelevinsky [FZIV]. We will prove the following theorem, which expresses the function $\mathbb{I}_{\mathcal{D}}(l)$ in terms of these polynomials.



Theorem 1.2. Let l be a point of $\mathcal{D}_L(S, \mathbb{Z})$ as above. Then

$$\mathbb{I}_{\mathcal{D}}(l) = \frac{\prod_{c \in \mathcal{C}^\circ} F_c(\hat{X}_1, \dots, \hat{X}_n)}{\prod_{c \in \mathcal{C}} F_c(X_1, \dots, X_n)} B_1^{g_{l,1}} \dots B_n^{g_{l,n}} X_1^{h_{l,1}} \dots X_n^{h_{l,n}}$$

where the F_c are polynomials, the $g_{l,i}$ are integers, the $h_{l,i}$ are half integers, and the \hat{X}_i are

given by

$$\widehat{X}_i = X_i \prod_j B_j^{\varepsilon_{ij}}.$$

All of the notation appearing in this theorem will be defined precisely below. As we will see, the above formula is in a sense a generalization of Corollary 6.3 in [FZIV]. If c is a closed loop belonging to \mathcal{C} or \mathcal{C}° , then F_c does not arise as one of Fomin and Zelevinsky's F -polynomials, but Theorem 1.2 suggests that the F_c should be viewed as “generalized” F -polynomials. Such generalized F -polynomials have appeared previously in the work of Musiker, Schiffler, and Williams [MW, MSW2] on cluster algebras associated to surfaces.

Because of the factors of $X_i^{1/2}$ in the above result, the function $\mathbb{I}_{\mathcal{D}}(l)$ is in general not expressible as a rational function in the B_i and X_i . In addition to proving the above result, we will show that the canonical mapping provides rational functions for a certain class of \mathcal{D} -laminations.

Theorem 1.3. Let l be a \mathcal{D} -lamination. Then $\mathbb{I}_{\mathcal{D}}(l)$ is a rational function in the variables B_i and X_i if and only if l has integral coordinates.

In fact, we will prove a more detailed statement which also gives a homological condition for the coordinates b_i and x_i of the lamination l to be integers.

In the final part of this paper, we discuss a map

$$\mathcal{I}_{\mathcal{D}} : \mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}_L(S, \mathbb{Z}) \rightarrow \mathbb{Q}$$

which we call the *intersection pairing*. Roughly speaking, if l and m are \mathcal{D} -laminations, we let k be half the minimal number of intersections between the portions of l and m that lie on S , we let k° be half the minimal number of intersections between the portions that lie on S° , and we define $\mathcal{I}_{\mathcal{D}}(l, m) = k^\circ - k$. In the final part of this paper, we will define this pairing more precisely and prove that it is the tropicalization of $\mathbb{I}_{\mathcal{D}}$ in an appropriate sense.

1.3 Organization

In Section 2, we review the results that we will need from [FG1, FG2, FG3, P], modifying the presentation to suit our purposes. We begin with a discussion of Teichmüller and lamination spaces and construct coordinates on these spaces. We explain how the properties of these coordinates lead to the definition of cluster varieties, and we explain the sense in which the Teichmüller and lamination spaces are dual.

In Section 3, we review the results of [FG4, FG6] and define the space $\mathcal{D}^+(S)$. We construct coordinates on this space and derive formulas for transforming between different coordinate systems. We then introduce the symplectic double and describe the problem we wish to solve.

In Section 4, we define \mathcal{D} -laminations on the double of a surface. We construct coordinates on the space of all \mathcal{D} -laminations and derive formulas for transforming between different coordinate systems. We will see that the space of \mathcal{D} -laminations is a tropical version of the

symplectic double. We introduce the space of real \mathcal{D} -laminations and describe a boundary for the Teichmüller \mathcal{D} -space using results of [FG5].

In Section 5, we review some results on cluster algebras and F -polynomials that play a role in our discussion of the canonical pairing. The references for this section are [FZIV, FST, MSW1]. In addition, we introduce a particular cluster algebra that we call the *cluster \mathcal{D} -algebra*.

Finally, in Section 6, we define a canonical pairing of \mathcal{D} -laminations with points of the Teichmüller \mathcal{D} -space. Our main result in this section is a formula for this pairing in terms of F -polynomials. We show that any lamination with integral coordinates provides a rational function on $\mathcal{D}^+(S)$. We conclude by defining a map $\mathcal{I}_{\mathcal{D}} : \mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}_L(S, \mathbb{Z}) \rightarrow \mathbb{Q}$ which we call the *intersection pairing*. We show that it is the tropicalization of the canonical pairing $\mathbb{I}_{\mathcal{D}}$ in an appropriate sense.

2 Background on cluster varieties

2.1 Teichmüller spaces

2.1.1 Preliminaries on surfaces

In this section, we review the basic theory of cluster varieties [FG1, FG2, FG3]. We begin by reviewing some geometric constructions on surfaces.

A *decorated surface* is a smooth oriented surface of genus $g \geq 0$ with $r \geq 0$ punctures and $s \geq 0$ smooth boundary components, where each smooth boundary component has finitely many marked points.

Definition 2.1. Let S be a decorated surface. An *ideal triangulation* T of S is a triangulation whose vertices are the marked points and the punctures.

From now on, we will consider only decorated surfaces S that admit an ideal triangulation. Note that in general the sides of a triangle in an ideal triangulation may not be distinct. In this case, the triangle is said to be *self-folded*.

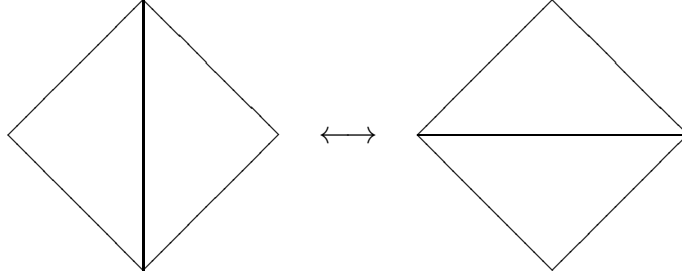
An edge of an ideal triangulation T is called *external* if it lies along the boundary of S , connecting two marked points, and is called *internal* otherwise. We will write J for the set of internal edges of the triangulation T and I for the set of all edges. For a triangulation without self-folded triangles, we define a skew-symmetric matrix ε_{ij} ($i, j \in I$) by the formula

$$\varepsilon_{ij} = \sum_{t \in T} \langle i, t, j \rangle$$

where $\langle i, t, j \rangle$ equals $+1$ (respectively, -1) if i and j are sides of the triangle t and i lies in the counterclockwise (respectively, clockwise) direction from j with respect to their common vertex. Otherwise, we set $\langle i, t, j \rangle = 0$.

If k is an internal edge of the ideal triangulation T , then a *flip* at k is the transformation of T that removes the edge k and replaces it by the unique different edge that, together with

the remaining edges, forms a new ideal triangulation:



A flip will be called *regular* if none of the triangles above is self-folded. It is a fact that any two isotopy classes of ideal triangulations on a surface are related by a sequence of flips. Note that there is a natural bijection between the edges of an ideal triangulation and the edges of the triangulation obtained by a flip at some edge. If we use this bijection to identify edges of the flipped triangulation with the set I , then it is straightforward to show that a flip at an edge k of an ideal triangulation changes the matrix ε_{ij} to the matrix

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

2.1.2 Teichmüller \mathcal{A} - and \mathcal{X} -spaces

Recall that the classical *Teichmüller space* $\mathcal{T}(S)$ of a punctured surface S without marked points can be viewed as the quotient

$$\text{Hom}'(\pi_1(S), PSL(2, \mathbb{R})) / PSL(2, \mathbb{R})$$

where $\text{Hom}'(\pi_1(S), PSL(2, \mathbb{R}))$ is the set of all discrete and faithful representations of $\pi_1(S)$ into $PSL(2, \mathbb{R})$ such that the image of a loop surrounding a puncture is parabolic. The group $PSL(2, \mathbb{R})$ acts on this set by conjugation.

Let $\rho : \pi_1(S) \rightarrow PSL(2, \mathbb{R})$ be an element of the set described above. Then we can represent S as a quotient

$$S = \mathbb{H} / \Delta$$

where \mathbb{H} is the upper half plane and $\Delta = \rho(\pi_1(S))$ is a discrete subgroup of $PSL(2, \mathbb{R})$. By definition, the map ρ takes any loop surrounding a puncture to a parabolic transformation. If $x \in \partial\mathbb{H}$ is the fixed point of the parabolic transformation corresponding to a puncture p in S , then a horocycle in \mathbb{H} centered at x projects to a curve on S which we also call a *horocycle* at p .

Definition 2.2. If S is a decorated surface with no marked points, then we define the *Teichmüller \mathcal{A} -space* $\mathcal{A}^+(S)$ to be the space that parametrizes pairs (ρ, \mathcal{S}) where ρ is a point of $\mathcal{T}(S)$ and \mathcal{S} is a set of horocycles, one at each puncture.

More generally, suppose that S is any decorated surface. Delete the marked points on the boundary of S and double the resulting surface along its boundary arcs. This produces a punctured surface S' where each marked point in the original surface gives rise to a puncture in S' . The doubled surface S' comes equipped with a natural involution $\iota : S' \rightarrow S'$.

Definition 2.3. The *Teichmüller \mathcal{A} -space* $\mathcal{A}^+(S)$ is defined as the ι -invariant subspace of $\mathcal{A}^+(S')$.

Note that this space can be identified with the one defined previously in the special case where there are no marked points on S . We write $\mathcal{A}_0^+(S)$ for the set of points in $\mathcal{A}^+(S)$ such that if e is the segment of ∂S between two marked points, then the horocycles at the ends of e are tangent. When there is no possibility of confusion, we will simply write \mathcal{A}^+ and \mathcal{A}_0^+ .

The other version of Teichmüller space that we will consider parametrizes more general surface group representations. More precisely, we consider the set

$$\text{Hom}''(\pi_1(S), PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$$

where $\text{Hom}''(\pi_1(S), PSL(2, \mathbb{R}))$ is the set of all discrete and faithful representations of $\pi_1(S)$ into $PSL(2, \mathbb{R})$ such that the image of a loop surrounding a puncture is either parabolic or hyperbolic. The group $PSL(2, \mathbb{R})$ again acts by conjugation.

Suppose we are given a representation $\rho : \pi_1(S) \rightarrow PSL(2, \mathbb{R})$ in the set described above. A puncture p in the surface S will be called a *hole* if ρ maps the homotopy class of a loop surrounding p to a hyperbolic transformation.

Definition 2.4. If S is a decorated surface with no marked points, then we define the *Teichmüller \mathcal{X} -space* $\mathcal{X}^+(S)$ to be the space that parametrizes pairs (ρ, \mathcal{S}) where ρ is an element of the above quotient and \mathcal{S} is a set of orientations, one for each hole.

More generally, if S is any decorated surface, then by doubling S , we can construct the punctured surface S' with the natural involution $\iota : S' \rightarrow S'$ as before. One then has the following definition.

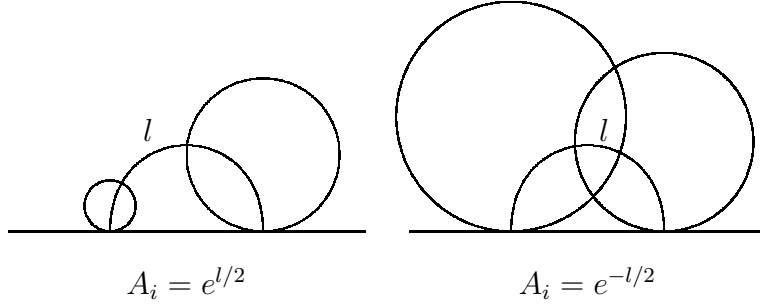
Definition 2.5. The *Teichmüller \mathcal{X} -space* $\mathcal{X}^+(S)$ of S is the ι -invariant subspace of $\mathcal{X}^+(S')$.

This space can be identified with the one defined previously in the special case where there are no marked points on S . When there is no possibility of confusion, we will simply write \mathcal{X}^+ .

2.1.3 Construction of coordinates

To construct coordinates on the Teichmüller \mathcal{A} -space, fix a point $m \in \mathcal{A}^+$ and let i be an edge of an ideal triangulation. The point m allows us to write S' as a quotient $S' = \mathbb{H}/\Delta$ where Δ is a discrete subgroup of $PSL(2, \mathbb{R})$. We can then deform i into a geodesic and lift this geodesic to the upper half plane \mathbb{H} . By definition of the Teichmüller \mathcal{A} -space, we have horocycles at the ends of the resulting geodesic in \mathbb{H} . We define A_i as the exponentiated half length (respectively, negative half length) of the segment of the lifted curve between the

intersection points with the horocycles if these horocycles do not intersect (respectively, if they do intersect).



Doing this for every edge of an ideal triangulation of S , we get a collection of numbers A_i ($i \in I$) corresponding to the point m .

Proposition 2.6. The numbers A_i ($i \in I$) provide a bijection

$$\mathcal{A}^+(S) \rightarrow \mathbb{R}_{>0}^{|I|}.$$

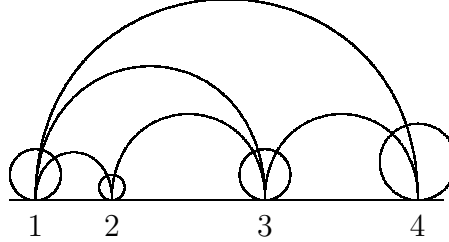
Proof. We will construct an inverse to this map. Let us suppose that we are given a positive number A_i for each edge $i \in I$. Let \tilde{S} denote the topological universal cover of S . Then we can lift the ideal triangulation of S to a triangulation of \tilde{S} , and we can associate to each edge of this triangulation the number associated to its projection.

Let t_0 be any triangle in the triangulation of \tilde{S} . By [P], Chapter 1, Corollary 4.8, there exists an ideal triangle u_0 in \mathbb{H} and horocycles around the endpoints of u_0 realizing the A -coordinates associated to the edges of t_0 . Next, consider a triangle t adjacent to t_0 in the triangulation of \tilde{S} . The common edge $t \cap t_0$ corresponds to an edge in \mathbb{H} with horocycles around its endpoints. By [P], Chapter 1, Lemma 4.14, there is a unique ideal triangle u in \mathbb{H} adjacent to u_0 with horocycles around its endpoints so that these horocycles agree with the ones already constructed and realize the A -coordinates associated to the edges of t .

Continuing in this way, we obtain a collection of ideal triangles in \mathbb{H} where each triangle corresponds to a triangle in \tilde{S} . Now any element $\gamma \in \pi_1(S)$ corresponds to a deck transformation of \tilde{S} , and there is a unique element of $PSL(2, \mathbb{R})$ that realizes this deck transformation as an isometry of \mathbb{H} preserving the triangulation. In this way, we obtain a representation $\rho : \pi_1(S) \rightarrow PSL(2, \mathbb{R})$. One can check that this construction provides a two-sided inverse of the map $\mathcal{A}^+(S) \rightarrow \mathbb{R}_{>0}^{|I|}$. \square

Now consider a point $m \in \mathcal{X}^+$ and an ideal triangulation T . Modify the neighborhood of each hole in S to get a new surface S' with geodesic boundary. The edges of T correspond to arcs on this surface S' , and there is a canonical way to deform these arcs. If an arc ends on a hole, we wind its endpoint around the hole infinitely many times in the direction prescribed by the orientation so that the arcs spiral into the holes in S' . More precisely, each geodesic boundary component of S' lifts to a geodesic in \mathbb{H} and we deform the preimage of an edge by dragging its endpoints along these geodesics until they coincide with points of $\partial\mathbb{H}$.

Once we have deformed the edges of our triangulation in this way, we can lift the triangulation to the upper half plane to get a collection of ideal triangles. Let k be any internal edge and consider the two ideal triangles that share this edge. Together they form an ideal quadrilateral, and we number the vertices of this ideal quadrilateral in counterclockwise order as shown below so that k joins vertices 1 and 3.



Choose a horocycle at each vertex, and put $A_{ij} = e^{l_{ij}/2}$ where l_{ij} is the signed length of the segment between the horocycles at i and j . We then define the *cross ratio*

$$X_k = \frac{A_{12}A_{34}}{A_{14}A_{23}}.$$

It is easy to see that there are two ways of numbering the vertices, and both give the same value for the cross ratio. One can also show that the numbers X_k ($k \in J$) are independent of the chosen horocycles. They are the numbers that we associate to the point m .

Proposition 2.7. The numbers X_j ($j \in J$) provide a bijection

$$\mathcal{X}^+(S) \rightarrow \mathbb{R}_{>0}^{|J|}.$$

Proof. Suppose we are given a positive number X_j for each edge $j \in J$. To recover the orientation of a hole h , we compute the number $\sum \log X_j$, where the sum is over all edges j incident to h . This sum is negative (respectively, positive) when the orientation is induced from the orientation of S (respectively, the opposite of this orientation).

Let \tilde{S} denote the topological universal cover of the surface S . We can lift the ideal triangulation of S to an ideal triangulation of \tilde{S} , and we can associate to each edge of this triangulation the number associated to its projection.

Let t_0 be any triangle in the triangulation of \tilde{S} , and choose a corresponding ideal triangle u_0 in \mathbb{H} . Next, consider a triangle t adjacent to t_0 in the triangulation of \tilde{S} . There is an X -coordinate associated to the common edge $t \cap t_0$. Since the cross ratio is a complete invariant of four points on $\partial\mathbb{H}$, there is a unique ideal triangle u in \mathbb{H} adjacent to u_0 so that these triangles realize the X -coordinate associated to the common edge.

Continuing in this way, we obtain a triangulation of a region in \mathbb{H} by geodesic triangles where each triangle corresponds to a triangle in \tilde{S} . Now any element $\gamma \in \pi_1(S)$ determines a deck transformation of \tilde{S} , and there is a unique element of $PSL_2(\mathbb{R})$ that realizes this deck transformation as an isometry of \mathbb{H} preserving the triangulation. In this way, we obtain a representation $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{R})$. One can check that this construction provides a two-sided inverse of the map $\mathcal{X}^+(S) \rightarrow \mathbb{R}_{>0}^{|J|}$. \square

One can show that a regular flip at an edge k of the ideal triangulation changes the coordinates A_i and X_i to new coordinates A'_i and X'_i given by the formulas

$$A'_i = \begin{cases} A_k^{-1} \left(\prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}} \right) & \text{if } i = k \\ A_i & \text{if } i \neq k \end{cases}$$

and

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i (1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k. \end{cases}$$

The canonical map $p : \mathcal{A}^+ \rightarrow \mathcal{X}^+$ which forgets the horocycles is given in terms of coordinates and the matrix ε_{ij} by the formula $p^*(X_i) = \prod_{j \in I} A_j^{\varepsilon_{ij}}$.

2.2 Lamination spaces

2.2.1 \mathcal{A} - and \mathcal{X} -laminations

Let S be a decorated surface. By a *curve* on S , we mean an embedding $C \rightarrow S$ of a compact, connected, one-dimensional manifold C with (possibly empty) boundary into S . We require that any endpoints of C map to punctures or points on the boundary of S disjoint from the marked points. When we talk about homotopies, we mean homotopies within the class of such curves. A curve is called *special* if it is retractable to a puncture or to an interval on ∂S containing exactly one marked point. A curve is *contractible* if it can be retracted to a point within this class of curves.

Definition 2.8. A *rational \mathcal{A} -lamination* on S is the homotopy class of a collection of finitely many nonintersecting noncontractible curves on S , either closed or ending on a segment of the boundary bounded by adjacent marked points, with rational weights and subject to the following conditions and equivalence relations:

1. The weight of a curve is nonnegative unless the curve is special.
2. A lamination containing a curve of weight zero is equivalent to the lamination with this curve removed.
3. A lamination containing homotopic curves of weights a and b is equivalent to the lamination with one curve removed and the weight $a + b$ on the other.

The set of all rational \mathcal{A} -laminations on S is denoted $\mathcal{A}_L(S, \mathbb{Q})$. We will write $\mathcal{A}_L(S, \mathbb{Z})$ for the set of all \mathcal{A} -laminations on S that can be represented by a collection of curves with integral weights. We write $\mathcal{A}_L^0(S, \mathbb{Z})$ for the subset of all laminations such that if e is the segment of ∂S between two marked points, then the total weight of the curves ending on e vanishes. When there is no possibility of confusion, we will simply write \mathcal{A}_L and \mathcal{A}_L^0 .

Definition 2.9. A *rational \mathcal{X} -lamination* on S is the homotopy class of a collection of finitely many nonintersecting noncontractible and non-special curves on S with positive rational weights and a choice of orientation for each puncture in S that meets a curve. A lamination containing homotopic curves of weights a and b is equivalent to the lamination with one curve removed and the weight $a + b$ on the other.

The set of all rational \mathcal{X} -laminations on S is denoted $\mathcal{X}_L(S, \mathbb{Q})$. We will write $\mathcal{X}_L(S, \mathbb{Z})$ for the set of all \mathcal{X} -laminations on S that can be represented by a collection of curves with integral weights. When there is no possibility of confusion, we will simply write \mathcal{X}_L .

2.2.2 Construction of coordinates

To construct coordinates on $\mathcal{A}_L(S, \mathbb{Q})$, fix an \mathcal{A} -lamination l and an ideal triangulation of S . Deform the curves of l so that each curve intersects each edge of the triangulation in the minimal number of points. Then we define a_i to be half the total weight of curves that intersect the edge i .

Proposition 2.10. The numbers a_i ($i \in I$) provide a bijection

$$\mathcal{A}_L(S, \mathbb{Q}) \rightarrow \mathbb{Q}^{|I|}.$$

Proof. Suppose we are given a rational number a_i for each edge $i \in I$. It is enough to construct an \mathcal{A} -lamination with coordinates

$$\tilde{a}_i = pa_i + q$$

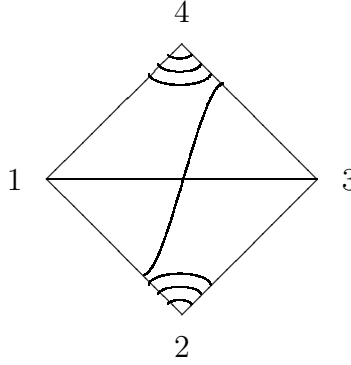
for some rational numbers p and q . Indeed, if we construct a lamination with these coordinates, then we can add a special curve of weight $-q$ around each puncture and each marked point and then divide the weight of each curve by p to get a lamination with coordinates a_i .

Let \tilde{S} denote the universal cover of S . Then we can lift the triangulation of S to a triangulation of \tilde{S} , and we can associate to each edge of this triangulation the number \tilde{a}_i associated to its projection.

Let t be any triangle in the triangulation of \tilde{S} . By an argument in [FG3], we can fix p and q so that there exists a corresponding topological triangle u with finitely many nonintersecting curves joining each pair of adjacent edges where the edge corresponding to i intersects the curves exactly $2\tilde{a}_i$ times. By gluing the triangles u and matching the curves at each edge, we recover the universal cover of S together with a collection of curves. Quotienting the resulting space by the group of deck transformations, we obtain the desired \mathcal{A} -lamination on the surface S . \square

Now consider an \mathcal{X} -lamination l and an ideal triangulation T . Remove a small neighborhood of each puncture in S that meets a curve to get a new surface S' with boundary. The edges of T correspond to arcs on this surface S' . If an arc ends on a hole, we wind its endpoint around the hole infinitely many times in the direction prescribed by the orientation so that the arcs spiral into the holes in S' .

Let k be an internal edge of the resulting triangulation of S' , and consider the quadrilateral on the surface with diagonal k . There will be finitely many curves that connect opposite sides of the quadrilateral and possibly infinitely many curves that join adjacent sides. Number the vertices of this quadrilateral in counterclockwise order as shown below so that the edge k joins vertices 1 and 3.



Let p be a vertex of this quadrilateral. If there are infinitely many curves connecting the edges that meet at p , then we can choose one such curve α_p and delete all of the curves between α_p and the point p . By doing this for each vertex, we remove all but finitely many curves from the quadrilateral and get an \mathcal{A} -lamination on a disk with four marked points. Let a_{ij} be the coordinate of this \mathcal{A} -lamination corresponding to the edge connecting vertices i and j . Define

$$x_k = a_{12} + a_{34} - a_{14} - a_{23}.$$

It is easy to see that this number is independent of all choices made in the construction. It is the number associated to the edge k .

Proposition 2.11. The numbers x_j ($j \in J$) provide a bijection

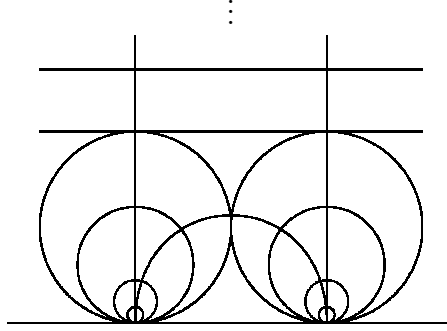
$$\mathcal{X}_L(S, \mathbb{Q}) \rightarrow \mathbb{Q}^{|J|}.$$

Proof. Suppose we are given a rational number x_j for each edge $j \in J$. To recover the orientation of a puncture h , we compute the number $\sum x_i$, where the sum is over all edges j incident to h . This sum is negative (respectively, positive) when the orientation is induced from the orientation of S (respectively, the opposite of this orientation).

Let \tilde{S} denote the universal cover of the surface S . We can lift the ideal triangulation of S to an ideal triangulation of \tilde{S} , and we can associate to each edge of this triangulation the number associated to its projection.

Let t_0 be any triangle in the triangulation of \tilde{S} , and choose a corresponding ideal triangle u_0 in the hyperbolic plane \mathbb{H} . There is a unique triple of horocycles about its endpoints that are pairwise tangent. The points of tangency provide three canonical points on the edges of u_0 . Parametrize the edges of this triangle by \mathbb{R} , respecting the orientation induced by the orientation of the triangle, so that the point with parameter $s \in \mathbb{R}$ lies at distance $|s|$ from the distinguished point. Connect points with parameter $s \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ on one side to points

with parameter $-s$ on the next side in the clockwise direction by a horocyclic arc. In this way, we obtain the triangle u_0 with infinitely many arcs connecting adjacent sides.



Next, consider a triangle t adjacent to t_0 in the ideal triangulation of \tilde{S} . There is a number x_j associated to the common edge j , and we can find an ideal triangle u adjacent to u_0 so that the cross ratio of the resulting quadrilateral is e^{x_j} . We can draw infinitely many horocyclic arcs on u as we did for u_0 . By [P], Chapter 1, Corollary 4.16, these arcs connect to the ones already drawn on u_0 .

Continuing in this way, we obtain a triangulation of a region in \mathbb{H} by ideal triangles where each triangle corresponds to a triangle in \tilde{S} . Quotienting this region by the action of the fundamental group, we obtain a surface homeomorphic to S with curves drawn on it. One can check that this construction provides a two-sided inverse of the map $\mathcal{X}_L(S) \rightarrow \mathbb{Q}^{|J|}$. \square

One can show that a regular flip at an edge k of the ideal triangulation changes the coordinates a_i and x_i to new coordinates a'_i and x'_i given by the formulas

$$a'_i = \begin{cases} \max\left(\sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj}a_j, -\sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj}a_j\right) - a_k & \text{if } i = k \\ a_i & \text{if } i \neq k \end{cases}$$

and

$$x'_i = \begin{cases} -x_k & \text{if } i = k \\ x_i + \varepsilon_{ki} \max(0, \operatorname{sgn}(\varepsilon_{ki})x_k) & \text{if } i \neq k. \end{cases}$$

Since these transformation rules are continuous with respect to the standard topologies on $\mathbb{Q}^{|I|}$ and $\mathbb{Q}^{|J|}$, the coordinates define natural topologies on the spaces of \mathcal{A} - and \mathcal{X} -laminations. We define the space of *real \mathcal{A} -laminations* or *real \mathcal{X} -laminations* as the metric space completion of the corresponding space of rational laminations.

The canonical map $p : \mathcal{A}_L \rightarrow \mathcal{X}_L$ which forgets the special curves is given in terms of coordinates and the matrix ε_{ij} by the formula $p^*(x_i) = \sum_{j \in I} \varepsilon_{ij}a_j$.

2.3 Cluster varieties

2.3.1 \mathcal{A} - and \mathcal{X} -varieties

We will now define the cluster \mathcal{A} - and \mathcal{X} -varieties. These are algebro-geometric objects defined using the transformation formulas given above.

Definition 2.12. A *seed* $\mathbf{i} = (I, J, \varepsilon_{ij}, d_i)$ consists of a finite set I , a subset $J \subseteq I$, a \mathbb{Q} -valued function ε_{ij} on $I \times I$ such that $\varepsilon_{ij} \in \mathbb{Z}$ if $i \in J$ or $j \in J$, and positive rational numbers d_i ($i \in I$) such that $\widehat{\varepsilon}_{ij} = \varepsilon_{ij}d_j^{-1}$ is skew-symmetric. The function ε_{ij} is called the *exchange function*, and the set $I - J$ is called the set of *frozen elements* of I .

Given a seed $\mathbf{i} = (I, J, \varepsilon_{ij}, d_i)$, we get two split algebraic tori $\mathcal{X}_{\mathbf{i}} = (\mathbb{G}_m)^{|J|}$ and $\mathcal{A}_{\mathbf{i}} = (\mathbb{G}_m)^{|I|}$ called the *seed \mathcal{X} -torus* and *seed \mathcal{A} -torus*, respectively. Let $\{X_j\}$ be the natural coordinates on the seed \mathcal{X} -torus and $\{A_i\}$ the natural coordinates on the seed \mathcal{A} -torus.

Definition 2.13. Let $\mathbf{i} = (I, J, \varepsilon_{ij}, d_i)$ be a seed and $k \in J$ a non-frozen element. Then we define a new seed $\mu_k(\mathbf{i}) = \mathbf{i}' = (I', J', \varepsilon'_{ij}, d'_i)$, called the seed obtained by *mutation* in the direction k by setting $I' = I$, $J' = J$, $d'_i = d_i$, and

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} + \frac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

A seed mutation also induces birational maps on the seed \mathcal{A} - and \mathcal{X} -tori defined by the formulas

$$\mu_k^* A'_i = \begin{cases} A_k^{-1} \left(\prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}} \right) & \text{if } i = k \\ A_i & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^* X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

where A'_i and X'_i are the coordinates on $\mathcal{A}_{\mathbf{i}'}$ and $\mathcal{X}_{\mathbf{i}'}$.

Two seeds will be called *mutation equivalent* if they are related by a sequence of mutations. We will denote the mutation equivalence class of a seed \mathbf{i} by $|\mathbf{i}|$. A transformation of the \mathcal{A} - or \mathcal{X} -tori obtained by composing the above birational maps is called a *cluster transformation*.

Definition 2.14. The *cluster \mathcal{A} -variety* $\mathcal{A} = \mathcal{A}_{|\mathbf{i}|}$ is a scheme over \mathbb{Z} obtained by gluing all seed \mathcal{A} -tori for seeds mutation equivalent to the seed \mathbf{i} using the above birational maps. The *cluster \mathcal{X} -variety* $\mathcal{X} = \mathcal{X}_{|\mathbf{i}|}$ is obtained by gluing all seed \mathcal{X} -tori for seeds mutation equivalent to the seed \mathbf{i} using the above birational maps.

There is a canonical mapping $p : \mathcal{A} \rightarrow \mathcal{X}$ given in any cluster coordinate system by $p^*(X_i) = \prod_{j \in I} A_j^{\varepsilon_{ij}}$.

2.3.2 Positive real and tropical points

We can now explain how the Teichmüller and lamination spaces arise as the positive real and tropical points of cluster varieties.

Definition 2.15. A *semifield* \mathbb{P} is a set equipped with binary operations $+$ and \cdot such that $+$ is commutative and associative, \mathbb{P} is an abelian group under \cdot , and the usual distributive law holds: $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in \mathbb{P}$.

Example 2.16. The following examples of a semifields will play an important role in our discussion.

1. Let $\mathbb{P} = \mathbb{Q}_{>0}$ or $\mathbb{P} = \mathbb{R}_{>0}$, the set of positive rational or real numbers. Then \mathbb{P} is a semifield under the usual operations of addition and multiplication.
2. Let $\mathbb{P} = \mathbb{Z}, \mathbb{Q}$, or \mathbb{R} . Then \mathbb{P} is a semifield with the operations \oplus and \otimes given by

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b$$

for all $a, b \in \mathbb{P}$. The semifields defined in this way are called *tropical semifields* in the works of Fock and Goncharov. They are denoted \mathbb{Z}^t , \mathbb{Q}^t , and \mathbb{R}^t .

3. Let $\mathbb{P} = \mathbb{Q}_{\text{sf}}(u_1, \dots, u_n)$ be the set of subtraction-free rational functions in the variables u_1, \dots, u_n . This set consists of all rational functions in u_1, \dots, u_n that are expressible as a ratio of two polynomials with positive integral coefficients. It is a semifield whose operations are ordinary addition and multiplication of rational functions.
4. Let $\mathbb{P} = \text{Trop}(y_1, \dots, y_n)$ be the free multiplicative abelian group generated by y_1, \dots, y_n with the auxiliary addition defined by

$$\prod_{i=1}^n y_i^{a_i} \oplus \prod_{i=1}^n y_i^{b_i} = \prod_{i=1}^n y_i^{\min(a_i, b_i)}.$$

This operation makes \mathbb{P} into a semifield which Fomin and Zelevinsky call the *tropical semifield* generated by y_1, \dots, y_n .

Given a semifield \mathbb{P} and a split algebraic torus H , we can form the set $H(\mathbb{P}) = X_*(H) \otimes_{\mathbb{Z}} \mathbb{P}$. Here $X_*(H)$ is the group of cocharacters of H and we are using the abelian group structure of \mathbb{P} . For example, we have the set $\mathcal{A}_i(\mathbb{P})$ obtained from the seed \mathcal{A} -torus \mathcal{A}_i . Now the maps $\psi_{i,i'} : \mathcal{A}_i \rightarrow \mathcal{A}_{i'}$ that we used to glue the seed \mathcal{A} -tori induce maps $\psi_{i,i'*} : \mathcal{A}_i(\mathbb{P}) \rightarrow \mathcal{A}_{i'}(\mathbb{P})$, so we can define the quotient

$$\mathcal{A}(\mathbb{P}) = \coprod \mathcal{A}_i(\mathbb{P}) / (\text{identifications } \psi_{i,i'*}).$$

The set of \mathbb{P} -points $\mathcal{X}(\mathbb{P})$ of the cluster \mathcal{X} -variety is defined similarly.

Since the coordinates of the Teichmüller \mathcal{A} - and \mathcal{X} -spaces are positive by construction, we see that the sets $\mathcal{A}(\mathbb{R}_{>0})$ and $\mathcal{X}(\mathbb{R}_{>0})$ are identified with the Teichmüller \mathcal{A} - and \mathcal{X} -spaces. By examining how the coordinates transform under mutation, we see also that the sets $\mathcal{A}(\mathbb{Q}^t)$ and $\mathcal{X}(\mathbb{Q}^t)$ are identified with the \mathcal{A} - and \mathcal{X} -lamination spaces.

One can show that $l \in \mathcal{X}_L(S, \mathbb{Z})$ if and only if l has integral coordinates, and hence

$$\mathcal{X}_L(S, \mathbb{Z}) = \mathcal{X}(\mathbb{Z}^t).$$

On the other hand, one can show that $l \in \mathcal{A}_L(S, \mathbb{Z})$ if and only if l has half integral coordinates and for any triangle the sum of the coordinates associated to its edges is an integer. Thus we have inclusions

$$\mathcal{A}(\mathbb{Z}^t) \subseteq \mathcal{A}_L(S, \mathbb{Z}) \subseteq \mathcal{A}\left(\frac{1}{2}\mathbb{Z}^t\right).$$

2.4 Canonical pairings

2.4.1 Multiplicative canonical pairings

As explained in [FG1] and [FG3], the Teichmüller and lamination spaces defined above are dual in the sense that there exists a multiplicative canonical pairing of either lamination space with the Teichmüller space of the opposite type. More precisely, we have a pair of maps

$$\begin{aligned} \mathbb{I} : \mathcal{X}_L(S, \mathbb{Z}) \times \mathcal{A}_0^+(S) &\rightarrow \mathbb{R}_{>0} \\ \mathbb{I} : \mathcal{A}_L^0(S, \mathbb{Z}) \times \mathcal{X}^+(S) &\rightarrow \mathbb{R}_{>0}, \end{aligned}$$

which we denote by the same symbol \mathbb{I} . We usually think of these maps as operations which assign to a lamination l the function on Teichmüller space given by $\mathbb{I}(l)(m) = \mathbb{I}(l, m)$.

Before giving the definition of the maps \mathbb{I} , let us describe two constructions involving Teichmüller spaces and laminations:

1. Let m be a point of \mathcal{A}_0^+ and l a curve on S connecting punctures or boundary segments on S . If an endpoint of l lies on a boundary segment, drag this endpoint in the counterclockwise direction until it hits a marked point. Take a geodesic homotopic to the resulting curve, and define $I(l, m)$ to be half the length between the horocycles at the endpoints of l .
2. Let m be a point of \mathcal{X}^+ and l a collection of curves with rational weights connecting boundary segments of S so that for any boundary segment, the total weight of the curves hitting it vanishes. If an endpoint lies on a boundary segment, drag this endpoint in the counterclockwise direction until it hits a marked point. Finally, take geodesics homotopic to the resulting curves. Choose a horocycle around every marked point. Then $I(l, m)$ is the weighted sum of the signed half lengths of the curve between the horocycles. One can check that this definition is independent of the choice of horocycles.

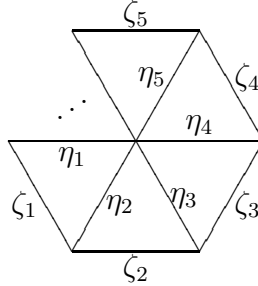
One now has the following definition of the canonical pairings.

Definition 2.17.

1. Let $l \in \mathcal{X}_L(S, \mathbb{Z})$ be a lamination consisting of a single closed curve of weight k , and let $m \in \mathcal{A}_0^+$. Then $\mathbb{I}(l, m)$ is the absolute value of the trace of the k th power of the monodromy around l .

2. Let $m \in \mathcal{X}^+$ and let $l \in \mathcal{A}_L^0(S, \mathbb{Z})$ be a lamination consisting of a single closed curve of weight k which is not retractable to a hole. Then $\mathbb{I}(l, m)$ is the absolute value of the trace of the k th power of the monodromy around l .
3. Let $m \in \mathcal{X}^+$ and let $l \in \mathcal{A}_L^0(S, \mathbb{Z})$ be a lamination consisting of a single closed curve of weight k which is retractable to a hole with positive (respectively, negative) orientation. Then $\mathbb{I}(l, m)$ is the absolute value of the smallest (respectively, largest) eigenvalue of the k th power of the monodromy around l .
4. Let $l \in \mathcal{X}_L(S, \mathbb{Z})$ be a lamination consisting of a single curve of weight k connecting punctures or points on the boundary of S , and let $m \in \mathcal{A}_0^+$. Assume that the orientation of any hole at one end of l is induced by the orientation of the surface. Then $\mathbb{I}(l, m) = \exp I(l, m)$.
5. Let $l \in \mathcal{A}_L^0(S, \mathbb{Z})$ be a lamination consisting of a collection of open curves of integral weights on S , and let $m \in \mathcal{X}^+$. Then $\mathbb{I}(l, m) = \exp I(l, m)$.
6. Let l_1 and l_2 be laminations such that no curve from l_1 intersects or is homotopic to a curve from l_2 . Then $\mathbb{I}(l_1 + l_2, m) = \mathbb{I}(l_1, m)\mathbb{I}(l_2, m)$.

This defines the canonical pairing in the special case where the orientations of the holes in an \mathcal{X} -lamination are induced from the orientation of the surface. If there is a hole p in an \mathcal{X} -lamination l whose orientation disagrees with the orientation of S , then we can define $\mathbb{I}(l)$ by modifying slightly the above definition. To do this, we first note that there is a natural invariant associated to p . Indeed, fix an ideal triangulation T of S and consider the triangles having p as a vertex. We can label the edges that meet p by η_1, \dots, η_N in counterclockwise order and we can write ζ_i for the third edge of the triangle having edges η_i and η_{i+1} , where we count indices modulo N .



Given a point $m \in \mathcal{A}_0^+(S)$, we get a number A_i associated to each edge i of T . We define

$$\alpha(p) = \sum_{i=1}^N \frac{A_{\zeta_i}}{A_{\eta_i} A_{\eta_{i+1}}}.$$

One can check that $\alpha(p)$ is independent of the choice of ideal triangulation.

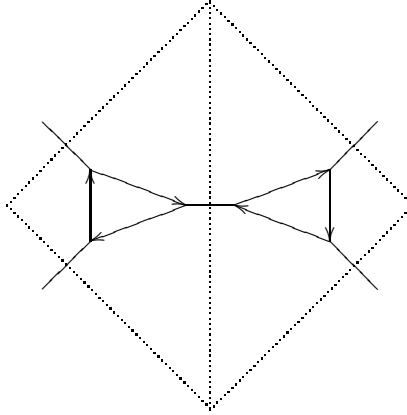
If r is the number of holes in the surface, then there is a natural action of the group $(\mathbb{Z}/2\mathbb{Z})^r$ on the Teichmüller \mathcal{A} -space [FG1]. The generator of the i th factor of $\mathbb{Z}/2\mathbb{Z}$ acts by multiplying A_j by $\alpha(p)$ whenever the edge j meets p . There is also an action of $(\mathbb{Z}/2\mathbb{Z})^r$ on

the space of \mathcal{X} -laminations where the generator of the i th factor of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the orientation of the i th hole. We can therefore extend the canonical mapping \mathbb{I} to all \mathcal{X} -laminations by requiring that it be equivariant with respect to these group actions.

2.4.2 Laurent polynomials from laminations

In [FG1] and [FG3], Fock and Goncharov describe a method for computing the functions $\mathbb{I}(l)$ in terms of the coordinates on the Teichmüller \mathcal{A} - or \mathcal{X} -space. Here we review the method in the case of the \mathcal{X} -space and discuss some of its implications.

To begin, let T be an ideal triangulation of the surface S . Draw a small edge transverse to every edge of the triangulation T and connect the endpoints of these edges pairwise within each triangle of T . In this way, we associate a graph Γ to the triangulation as illustrated below. (The original triangulation is indicated by the dotted lines in this picture.)



This construction produces a small triangle inside every triangle of T , and we orient the edges of these small triangles in the clockwise direction.

Suppose we are given an oriented closed loop γ based at a vertex of the graph. Then γ is homotopic to a closed path in Γ . Let e_1, \dots, e_n be the edges of this path in order, and let us associate a matrix $M(e_i)$ to each edge e_i as follows. If e_i intersects an edge k of T , then the matrix that we assign to this edge is

$$M(e_i) = \begin{pmatrix} 0 & X_k^{1/2} \\ -X_k^{-1/2} & 0 \end{pmatrix}.$$

This matrix satisfies $M(e_i)^2 = -1$ in $SL(2, \mathbb{R})$ and hence $M(e_i) = M(e_i)^{-1}$ in $PSL(2, \mathbb{R})$. On the other hand, if e_i does not intersect an edge of T and its orientation agrees with the orientation coming from γ , then we define

$$M(e_i) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Finally, if e_i does not intersect an edge of T and its orientation disagrees with the orientation of γ , then $M(e_i)$ is the inverse of this last matrix. We can multiply the matrices $M(e_i)$ defined

in this way to get the monodromy

$$\rho(\gamma) = M(e_n) \dots M(e_1) \in PSL(2, \mathbb{R}).$$

One can check that this element $\rho(\gamma)$ is well defined.

There is a similar construction that allows us to express the monodromy of a closed loop in terms of A -coordinates. In [FG1], Fock and Goncharov use these constructions to compute $\mathbb{I}(l)$ when l is an \mathcal{A} - or \mathcal{X} -lamination and to show that this function can be written as a Laurent polynomial with positive integral coefficients in the variables $X_j^{1/2}$ or A_j ($j \in J$), respectively. Moreover, they show that when l has integral coordinates (that is, when l comes from $\mathcal{A}(\mathbb{Z}^t)$ or $\mathcal{X}(\mathbb{Z}^t)$), the function $\mathbb{I}(l)$ can in fact be written as a Laurent polynomial with positive integral coefficients in the variables X_j or A_j ($j \in J$). In Section 6, we will revisit the above construction and use it to prove a similar result for \mathcal{D} -laminations.

2.4.3 The intersection pairing

In addition to the multiplicative canonical pairings that we defined above, we have the following canonical map, which should be viewed as a degeneration of these pairings.

Definition 2.18. Let S be a punctured surface and choose $l \in \mathcal{A}_L(S, \mathbb{Z})$ and $m \in \mathcal{X}_L(S, \mathbb{Z})$. Assume that m provides the negative orientation for each hole. Then we define $\mathcal{I}(l, m)$ to be half the minimal number of intersections between l and m . Here we take into account the weights of the curves so that if a curve of weight k_1 intersects a curve of weight k_2 , then this intersection contributes the term $k_1 k_2 / 2$ to $\mathcal{I}(l, m)$.

This defines $\mathcal{I}(l, m)$ in the special case where m provides the negative orientation for each hole. Note that there are natural actions of the group $(\mathbb{Z}/2\mathbb{Z})^r$ on $\mathcal{A}_L(S, \mathbb{Z})$ and $\mathcal{X}_L(S, \mathbb{Z})$ where r is the number of punctures in S . The generator of the i th factor of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{A}_L(S, \mathbb{Z})$ by changing the sign of the weight of curves surrounding the i th hole. It acts on $\mathcal{X}_L(S, \mathbb{Z})$ by changing the orientation of the i th hole. We extend \mathcal{I} to a map

$$\mathcal{I} : \mathcal{A}_L(S, \mathbb{Z}) \times \mathcal{X}_L(S, \mathbb{Z}) \rightarrow \mathbb{Q}$$

called the *intersection pairing* by requiring that it be equivariant with respect to these group actions.

To understand the relationship between the intersection pairing and the multiplicative canonical pairings defined above, we recall the notion of the tropicalization of a subtraction-free rational function [FG3]. Let $F(u_1, \dots, u_n)$ be such a function. We define its *tropicalization* $F^t(u_1, \dots, u_n)$ by the formula

$$F^t(u_1, \dots, u_n) = \lim_{C \rightarrow \infty} \frac{\log F(e^{Cu_1}, \dots, e^{Cu_n})}{C}.$$

It is easy to see that

$$\lim_{C \rightarrow \infty} \frac{\log(e^{Cv_1} + \dots + e^{Cv_n})}{C} = \max(v_1, \dots, v_n),$$

so tropicalization takes the operations $+$ and \cdot to \max and $+$, respectively.

For any laminations $l \in \mathcal{A}_L^0(S, \mathbb{Z})$ and $m \in \mathcal{X}_L(S, \mathbb{Z})$, one has

$$\mathcal{I}(l, m) = (\mathbb{I}(l))^t(m)$$

and

$$\mathcal{I}(l, m) = (\mathbb{I}(m))^t(l)$$

so that the intersection pairing is the tropicalization of the multiplicative canonical pairings. In Section 6, we will give a similar characterization of the tropicalization of the canonical mapping $\mathbb{I}_{\mathcal{D}}$ on \mathcal{D} -laminations.

3 The symplectic double

3.1 Teichmüller \mathcal{D} -space

We now describe the moduli space from [FG6]. Let Σ be an oriented punctured surface.

Definition 3.1. A *simple lamination* on Σ is a finite collection $\gamma = \{\gamma_i\}$ of simple nontrivial disjoint nonisotopic loops on Σ which do not retract to the punctures, considered modulo isotopy.

Let Σ be an oriented surface equipped with a simple lamination $\gamma = \{\gamma_i\}$. For any subset $\{\gamma_1, \dots, \gamma_k\}$ of the set of loops γ_i , we let Σ_{p_1, \dots, p_k} denote the singular surface obtained by pinching these loops to get the nodes p_1, \dots, p_k . This singular surface is equipped with a simple lamination γ_{p_1, \dots, p_k} given by the image of $\gamma - \{\gamma_1, \dots, \gamma_k\}$. Note that if we cut the surface Σ_{p_1, \dots, p_k} at the nodes p_1, \dots, p_k , we get a punctured surface $\Sigma'_{p_1, \dots, p_k}$. When we talk about a point in the Teichmüller space of Σ_{p_1, \dots, p_k} , we really mean a point in the Teichmüller space of the surface $\Sigma'_{p_1, \dots, p_k}$. Similarly, when we talk about a horocycle at a node of Σ_{p_1, \dots, p_k} , we really mean a horocycle at one of the corresponding punctures in $\Sigma'_{p_1, \dots, p_k}$.

Definition 3.2 ([FG6]). The set $\mathcal{X}_{\Sigma; \gamma; p_1, \dots, p_k}^+$ parametrizes points $\rho \in \mathcal{T}(\Sigma_{p_1, \dots, p_k})$ of the Teichmüller space of Σ_{p_1, \dots, p_k} , together with the following data:

1. An orientation for each loop of the simple lamination γ_{p_1, \dots, p_k} .
2. For every node p_i , a pair of horocycles $(c_{-,i}, c_{+,i})$ centered at p_i , located on opposite sides of the node, and defined up to simultaneous shift by any real number.

We write $\mathcal{X}_{\Sigma; \gamma}^+$ for the union of these sets $\mathcal{X}_{\Sigma; \gamma; p_1, \dots, p_k}^+$.

Now let S be a decorated surface. If S has punctures, then we can compactify S by gluing a boundary circle to each puncture. This produces a compact oriented surface with boundary which we will also denote by S . Let S° be the same surface with the opposite orientation.

The *double* $S_{\mathcal{D}}$ of S is the punctured surface obtained by gluing S and S° along corresponding boundary components and deleting the image of each marked point in the resulting surface. The surface $\Sigma = S_{\mathcal{D}}$ carries a natural simple lamination γ given by the image of the boundary loops of S .

Definition 3.3. The *Teichmüller \mathcal{D} -space* $\mathcal{D}^+(S)$ is the space $\mathcal{X}_{\Sigma, \gamma}^+$ with $\Sigma = S_{\mathcal{D}}$.

When there is no possibility of confusion, we will denote the Teichmüller \mathcal{D} -space by \mathcal{D}^+ .

3.2 Construction of coordinates

3.2.1 Preliminaries

Let S be a surface as above and $S_{\mathcal{D}}$ its double. The next two results will be used to construct coordinates on $\mathcal{D}^+(S)$.

Lemma 3.4. If $\rho \in \mathcal{T}(S_{\mathcal{D}})$ is a point in the Teichmüller space of $S_{\mathcal{D}}$, then there exists a hyperbolic structure representing ρ such that $\partial S \subseteq S_{\mathcal{D}}$ is geodesic.

Proof. A point ρ of the Teichmüller space can be viewed as a marked hyperbolic surface, that is, a hyperbolic surface X together with a diffeomorphism $\phi : S_{\mathcal{D}} \rightarrow X$. The image of $\partial S \subseteq S_{\mathcal{D}}$ under ϕ may not be geodesic in X , but we can deform ϕ to a homotopic map ϕ' such that $\phi'(\partial S)$ is geodesic. This is the same as saying that the diagram

$$\begin{array}{ccc} & S_{\mathcal{D}} & \\ \phi \swarrow & & \searrow \phi' \\ X & \xrightarrow{1_X} & X \end{array}$$

commutes up to homotopy. Thus (X, ϕ) and (X, ϕ') represent the same point of Teichmüller space. Pulling back the hyperbolic structure of X along ϕ' , we obtain a hyperbolic structure on $S_{\mathcal{D}}$ representing the point ρ . \square

Let $\rho \in \mathcal{T}(S_{\mathcal{D}})$. By the above lemma, we can view ρ as a hyperbolic structure on $S_{\mathcal{D}}$ such that $\partial S \subseteq S_{\mathcal{D}}$ is geodesic. By cutting along ∂S , we recover the surfaces S and S° equipped with hyperbolic structures with geodesic boundary. Then the universal cover \tilde{S} of S can be identified with a subset of \mathbb{H} obtained by removing countably many geodesic half disks. Similarly, the universal cover \tilde{S}° of S° can be identified with a subset of \mathbb{H} .

Lemma 3.5. Let ι be the natural map $S \rightarrow S^\circ$. Then there exists a map $\tilde{\iota} : \tilde{S} \rightarrow \tilde{S}^\circ$, unique up to the action of $\pi_1(S)$ by deck transformations, such that the diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\iota}} & \tilde{S}^\circ \\ \downarrow & & \downarrow \\ S & \xrightarrow{\iota} & S^\circ \end{array}$$

commutes. This map $\tilde{\iota}$ restricts to an isometry on the geodesic boundary of \tilde{S} .

Proof. Since \tilde{S} is simply connected, the composition $\tilde{S} \rightarrow S \xrightarrow{\iota} S^\circ$ induces the zero map on fundamental groups. Hence, by the lifting criterion, there exists a map $\tilde{\iota} : \tilde{S} \rightarrow \tilde{S}^\circ$ making the above diagram commutative. If $\tilde{\iota}' : \tilde{S} \rightarrow \tilde{S}^\circ$ is any other map with this property, then $\tilde{\iota}$ and $\tilde{\iota}'$ are both lifts of the composition $\tilde{S} \rightarrow S \xrightarrow{\iota} S^\circ$. It follows that they coincide up to a deck transformation.

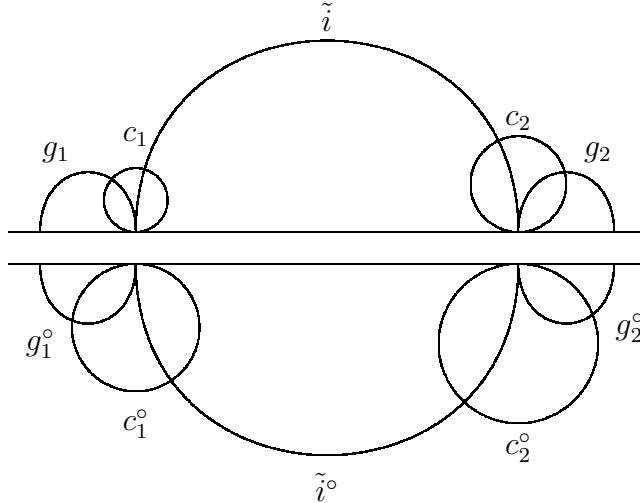
We know that the map $\tilde{\iota}$ restricts to an isometry on the geodesic boundary of \tilde{S} because the other three maps of the diagram restrict to isometries on boundary geodesics. \square

3.2.2 Definition of B_i and X_i

To construct coordinates on $\mathcal{D}^+(S)$, fix a point $m \in \mathcal{D}^+(S)$ and let $i \in J$ be an edge of the ideal triangulation T . The point m determines a point of the Teichmüller space of Σ_{p_1, \dots, p_k} where $\Sigma = S_{\mathcal{D}}$, and we can represent this by a hyperbolic structure such that the image of ∂S is geodesic. By cutting along the image of ∂S , we recover the surfaces S and S° equipped with hyperbolic structures with geodesic boundary.

Suppose the edge i corresponds to an arc connecting two holes in S . Choose a pair of geodesics g_1 and g_2 in the upper half plane \mathbb{H} which project to these boundary components. Deform the arc connecting the holes by winding its endpoints around the holes infinitely many times in the direction prescribed by the orientation. This corresponds to deforming the preimage \tilde{i} in \mathbb{H} so that its endpoints coincide with endpoints of g_1 and g_2 . If we let i° be the image of the arc i under the tautological map $S \rightarrow S^\circ$, then we can apply the same procedure to i° to get an arc \tilde{i}° in \mathbb{H} .

Choose horocycles c_1 and c_2 around the endpoints of \tilde{i} . The horocycle c_k intersects g_k in a unique point. By Lemma 3.5, there is a corresponding point on g_k° and thus a corresponding horocycle c_k° . The map constructed in Lemma 3.5 restricts to an isometry on g_k , so if we shift c_k by some amount, then the corresponding horocycle c_k° will shift by the same amount.



Denote by A_i the exponentiated signed half distance between c_1 and c_2 , and denote by A_i° the exponentiated signed half distance between c_1° and c_2° . We can then define a number

associated to the edge i by

$$B_i = \frac{A_i^\circ}{A_i}.$$

This defines B_i when i corresponds to an arc connecting two holes. If one or both of the endpoints of i are punctures, then the data of the point $m \in \mathcal{D}^+(S)$ provides a horocycle c_k at any endpoint of i which is a puncture, as well as a horocycle c_k° at the corresponding endpoint of i° . Thus we can associate numbers A_i and A_i° to these arc as before, and we can define B_i by the above formula. One can show that the $|J|$ numbers obtained in this way are independent of all choices made in the construction.

In addition to the B_i , there are $|J|$ numbers X_i associated to a point in the space $\mathcal{D}^+(S)$. Given a point of $\mathcal{D}^+(S)$, these are simply defined as the X -coordinates of the point of the Teichmüller \mathcal{X} -space of the surface S obtained by cutting $S_{\mathcal{D}}$ along the image of ∂S .

3.2.3 Reconstruction

We will now show that the B_i and X_i are indeed coordinates on $\mathcal{D}^+(S)$.

Proposition 3.6. The numbers B_i and X_i provide a bijection

$$\mathcal{D}^+(S) \rightarrow \mathbb{R}_{>0}^{2|J|}.$$

Proof. Call this map ϕ . We will construct a map $\psi : \mathbb{R}_{>0}^{2|J|} \rightarrow \mathcal{D}^+(S)$ and prove that ϕ and ψ are inverses.

Step 1. Definition of $\psi : \mathbb{R}_{>0}^{2|J|} \rightarrow \mathcal{D}^+(S)$.

Suppose we are given positive numbers B_j and X_j for every edge $j \in J$. By the reconstruction procedure for the Teichmüller \mathcal{X} -space, we can glue together ideal triangles in \mathbb{H} , using the X_i as gluing parameters, to get a tiling of a region $\tilde{S} \subseteq \mathbb{H}$ with geodesic boundary.

Choose a horocycle around each vertex of this triangulation. Then for every edge i , there is a number A_i defined as the exponentiated signed half length between the horocycles at the ends of i . If B_i is the number corresponding to the edge i , define

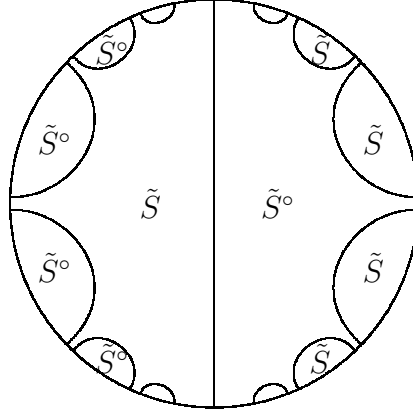
$$A_i^\circ = B_i A_i.$$

We will use these numbers A_i° to construct another region \tilde{S}° with geodesic boundary in a copy of \mathbb{H} .

To construct \tilde{S}° , first choose an edge i in the triangulation of \tilde{S} . Let i° be any geodesic in the second copy of \mathbb{H} . There are horocycles around the endpoints of i with corresponding exponentiated half length A_i , and we can choose horocycles around the endpoints of i° so that the resulting exponentiated half length equals A_i° . Consider one of the triangles adjacent to i . There are numbers A_j and A_k corresponding to its other sides. By [P], Chapter 1, Lemma 4.14, there is a unique point in the second copy of $\partial\mathbb{H}$ and a horocycle about this point so that the exponentiated half lengths of the decorated arcs incident to this point are A_j° and A_k° , and the cyclic order of the vertices of the resulting triangle agrees with the original cyclic order of the vertices.

Repeating this process for every edge in the triangulation of \tilde{S} , we obtain a region $\tilde{S}^\circ \subseteq \mathbb{H}$ with geodesic boundary. To construct this region, we had to choose horocycles around the endpoints of the triangulation of \tilde{S} , but the construction is independent of these choices. We also had to choose edges i and i° and horocycles around the endpoints of i° . If we had made a different choice we would get a different region obtained from \tilde{S}° by a transformation in $PSL(2, \mathbb{R})$. Thus \tilde{S}° is well defined up to isometry.

Now the region \tilde{S} is obtained from \mathbb{H} by removing infinitely many geodesic half disks. We can extend \tilde{S} to a larger region by gluing a copy of \tilde{S}° in each of these half disks in such a way that the horocycles around identified vertices coincide. The resulting region is again obtained from \mathbb{H} by removing infinitely many geodesic half disks, and we can enlarge it by gluing a copy of \tilde{S} in each of these half disks. Continuing this process ad infinitum, we partition the entire hyperbolic plane into ideal triangles. The illustration below shows an example of how the spaces \tilde{S} and \tilde{S}° may be glued together in the disk model of the hyperbolic plane.



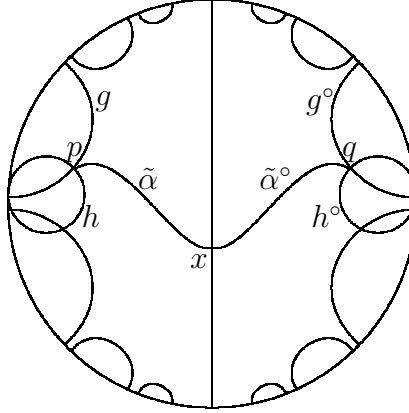
Any element of $\pi_1(S_{\mathcal{D}})$ corresponds to a deck transformation and is thus represented as a unique element of $PSL(2, \mathbb{R})$. In this way we recover a discrete and faithful representation $\rho : \pi_1(S_{\mathcal{D}}) \rightarrow PSL(2, \mathbb{R})$ representing a point in $\mathcal{T}(S_{\mathcal{D}})$.

Step 2. Proof of $\phi \circ \psi = 1_{\mathbb{R}_{>0}^{2|J|}}$.

Let B_j and X_j ($j \in J$) be given. By the reconstruction procedure described above, we construct triangulated regions $\tilde{S} \subseteq \mathbb{H}$ and $\tilde{S}^\circ \subseteq \mathbb{H}$ which we then glue together to get a triangulation of the hyperbolic plane. From this triangulation, we get a point $m \in \mathcal{D}^+(S)$. We want to show that the coordinates of m are the numbers B_j and X_j .

This point m determines a hyperbolic metric on $S_{\mathcal{D}}$, and the universal cover of the resulting hyperbolic surface is isometric to the triangulated surface $\tilde{S}_{\mathcal{D}}$ that we got by gluing together copies of \tilde{S} and \tilde{S}° . Consider a copy of \tilde{S} in $\tilde{S}_{\mathcal{D}}$. To find the coordinates of m , we choose horocycles around the vertices of the triangulation of \tilde{S} , and we can take these to be exactly the horocycles used in the construction of m from the B_j and X_j . Now consider a copy of \tilde{S}° adjacent to \tilde{S} in $\tilde{S}_{\mathcal{D}}$. Choose a basepoint x on the geodesic separating these regions. If g is any boundary geodesic of \tilde{S} , then there is a horocycle h around one of the endpoints of g , and this horocycle intersects g in a unique point p . Draw a curve $\tilde{\alpha}$ in \tilde{S}

from the point p to the point x .



This curve $\tilde{\alpha}$ projects to a curve α on the surface S , and we can apply the map ι to get a curve α° on S° . Finally, we lift α° to a curve $\tilde{\alpha}^\circ$ in S° that starts at x . This curve $\tilde{\alpha}^\circ$ ends at some point $q = \tilde{\iota}(p)$, and there is a horocycle h° that intersects g° at q . In this way, we draw horocycles around all of the vertices of \tilde{S}° and can define the coordinates of m .

On the other hand, consider the monodromy along the curve in $S_{\mathcal{D}}$ obtained by concatenating α and α° . It maps the region \tilde{S} isometrically into the region bounded by g° . The horocycle used in the construction of m from the B_j and X_j is obtained by applying this transformation to h . Thus the horocycles used to construct m agree with the ones obtained using the map $\tilde{\iota}$.

It follows that the B -coordinates of m are simply the numbers B_j that we started with. It is easy to see that the X -coordinates of m are the numbers X_j . This completes Step 2 of the proof.

Step 3. Proof of $\psi \circ \phi = 1_{\mathcal{D}^+}$.

Let $m \in \mathcal{D}^+(S)$ be given. To calculate the coordinates of m , let us pass to the universal cover $\tilde{S}_{\mathcal{D}}$ of $S_{\mathcal{D}}$ and choose a copy of \tilde{S} in $\tilde{S}_{\mathcal{D}}$. Choose horocycles around the vertices of the triangulation of this copy of \tilde{S} . Using the construction described above involving lifts of the map $\iota : S \rightarrow S^\circ$, we can get horocycles around all vertices of the triangulation of $S_{\mathcal{D}}$. We can use these horocycles to calculate the numbers A_j and A_j° , and thus the coordinates B_j and X_j .

We can now use the X_i to reconstruct the region \tilde{S} . We must choose horocycles around the vertices of the triangulation of \tilde{S} , and we can assume that these are exactly the ones used above to define the B_j and X_j . Then we can use the B_j to reconstruct an adjacent region \tilde{S}° with exactly the horocycles used above. By gluing together copies of \tilde{S} and \tilde{S}° , we recover $\tilde{S}_{\mathcal{D}}$ together with all of the horocycles used to compute the coordinates. Quotienting this universal cover by the action of $\pi_1(S_{\mathcal{D}})$, we recover the point $m \in \mathcal{D}^+(S)$. This completes Step 3 of the proof. \square

3.2.4 Transformation rule

We have defined the B_i and X_i coordinates in terms of a fixed ideal triangulation of S . The next result says how these coordinates vary when we change the triangulation.

Proposition 3.7. A regular flip at an edge k of the triangulation changes the coordinates X_i and B_i to new coordinates X'_i and B'_i given by the formulas

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

$$B'_i = \begin{cases} \frac{X_k \prod_{j|\varepsilon_{kj}>0} B_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} B_j^{-\varepsilon_{kj}}}{(1+X_k)B_k} & \text{if } i = k \\ B_i & \text{if } i \neq k. \end{cases}$$

Proof. By definition, the coordinates X_i on $\mathcal{D}^+(S)$ coincide with the coordinates of the corresponding point of the Teichmüller \mathcal{X} -space of S . Therefore the transformation rule for the X_i is just the usual transformation rule for the coordinates on the \mathcal{X} -space.

To prove the second rule, deform the arcs corresponding to the edges of the ideal triangulation by winding their endpoints around the curves of the simple lamination infinitely many times. Lift these deformed curves to the universal cover of S , and consider the ideal quadrilateral formed by the two triangles adjacent to the geodesic arc \tilde{k} corresponding to k .

Number the vertices of this quadrilateral in counterclockwise order so that the edge \tilde{k} joins vertices 1 and 3. Choose a horocycle around each vertex of this ideal quadrilateral and let A_{ij} denote the exponentiated signed half length between the horocycles at i and j . There is a corresponding ideal quadrilateral in the universal cover of S° and a number A_{ij}° corresponding to the edge connecting vertices i and j of this ideal quadrilateral.

If we flip at the edge k , then the ideal quadrilateral in the universal cover of S is replaced by the same ideal quadrilateral triangulated by the arc from 2 to 4, and the number associated to this new arc is

$$A_{24} = \frac{A_{12}A_{34} + A_{14}A_{23}}{A_{13}}.$$

Similarly, the number A_{13}° transforms to

$$A_{24}^\circ = \frac{A_{12}^\circ A_{34}^\circ + A_{14}^\circ A_{23}^\circ}{A_{13}^\circ}.$$

Applying these rules to the quotient $B_{13} = A_{13}^\circ/A_{13}$, we obtain

$$\begin{aligned} B_{24} &= \frac{A_{12}^\circ A_{34}^\circ + A_{14}^\circ A_{23}^\circ}{A_{12}A_{34} + A_{14}A_{23}} \frac{A_{13}}{A_{13}^\circ} \\ &= \frac{\frac{A_{12}^\circ A_{34}^\circ}{A_{14}A_{23}} + \frac{A_{14}^\circ A_{23}^\circ}{A_{14}A_{23}}}{1 + \frac{A_{12}A_{34}}{A_{14}A_{23}}} \frac{A_{13}}{A_{13}^\circ} \\ &= \frac{X_{13}B_{12}B_{34} + B_{14}B_{23}}{(1 + X_{13})B_{13}} \end{aligned}$$

where we have used the relation $X_{13} = \frac{A_{12}A_{34}}{A_{14}A_{23}}$. This proves the transformation rule for the B_i . \square

3.3 Definition of the symplectic double

We will now use the transformation rules stated above to define the symplectic double. For any seed $\mathbf{i} = (I, J, \varepsilon_{ij}, d_i)$ with $I = J$, we define a split algebraic torus $\mathcal{D}_{\mathbf{i}} = (\mathbb{G}_m)^{2|J|}$ with natural coordinates $\{B_i, X_i\}_{i \in J}$. A seed mutation induces a map on tori defined by

$$\mu_k^* X'_i = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

and

$$\mu_k^* B'_i = \begin{cases} \frac{X_k \prod_{j|\varepsilon_{kj}>0} B_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} B_j^{-\varepsilon_{kj}}}{(1+X_k)B_k} & \text{if } i = k \\ B_i & \text{if } i \neq k \end{cases}$$

where X'_i and B'_i are the coordinates on $\mathcal{D}_{\mathbf{i}'}$.

Definition 3.8. The *symplectic double* $\mathcal{D} = \mathcal{D}_{|\mathbf{i}|}$ is a scheme over \mathbb{Z} obtained by gluing the \mathcal{D} -tori for all seeds mutation equivalent to the seed \mathbf{i} using the above birational maps.

Our goal in the next section is to describe a space of laminations which is related to the symplectic double in the same way that the spaces of \mathcal{A} - and \mathcal{X} -laminations are related to the cluster \mathcal{A} - and \mathcal{X} -varieties. These new laminations will be called *\mathcal{D} -laminations*.

The space $\mathcal{D}_L(S, \mathbb{Q})$ of rational \mathcal{D} -laminations should be a tropical version of the symplectic double in the sense that it is identified with $\mathcal{D}(\mathbb{Q}^t)$. More concretely, this means that there are coordinates $\{b_i, x_i\}_{i \in J}$ associated to the edges of an ideal triangulation of S , and if we flip the triangulation at an edge k , we get new coordinates

$$x'_i = \begin{cases} -x_k & \text{if } i = k \\ x_i + \varepsilon_{ki} \max(0, \text{sgn}(\varepsilon_{ki})x_k) & \text{if } i \neq k \end{cases}$$

and

$$b'_i = \begin{cases} \max\left(x_k + \sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj}b_j, -\sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj}b_j\right) - \max(0, x_k) - b_k & \text{if } i = k \\ b_i & \text{if } i \neq k. \end{cases}$$

4 \mathcal{D} -laminations

4.1 The notion of a \mathcal{D} -lamination

Let S be a compact oriented surface with finitely many marked points on its boundary, and let S° be the same surface with the opposite orientation. Recall that the *double* $S_{\mathcal{D}}$

of S is defined as the surface obtained by gluing S and S° along corresponding boundary components.

By a (closed) *curve* on $S_{\mathcal{D}}$, we mean an embedding of a circle into $S_{\mathcal{D}}$ whose image is disjoint from the marked points. When we talk about homotopies, we mean homotopies within the class of such curves. A curve is called *contractible* if it can be retracted to a point within this class of curves. It is *special* if it is retractible to a marked point.

As before, a *simple lamination* on such a surface is a finite collection $\gamma = \{\gamma_i\}$ of simple noncontractible nonspecial disjoint nonisotopic loops considered up to isotopy. The surface $S_{\mathcal{D}}$ comes equipped with a simple lamination γ given by the image of the boundary loops of S in $S_{\mathcal{D}}$.

Definition 4.1. A *rational \mathcal{D} -lamination* on $S_{\mathcal{D}}$ is the homotopy class of a collection of finitely many nonintersecting, noncontractible, and non-special closed curves with positive rational weights and a choice of orientation for each component of γ which meets or is homotopic to a curve. A lamination containing homotopic curves of weights a and b is equivalent to the lamination with one curve removed and the weight $a + b$ on the other.

The set of all rational \mathcal{D} -laminations on $S_{\mathcal{D}}$ will be denoted $\mathcal{D}_L(S, \mathbb{Q})$. We will write $\mathcal{D}_L(S, \mathbb{Z})$ for the set of all \mathcal{D} -laminations on $S_{\mathcal{D}}$ that can be represented by a collection of curves with integral weights. When there is no possibility of confusion, we will simply write \mathcal{D}_L .

4.2 Construction of coordinates

4.2.1 Preliminaries

Let $l \in \mathcal{D}_L(S, \mathbb{Q})$. Then l can be represented by a collection of nonintersecting simple closed curves on $S_{\mathcal{D}}$ such that all curves have the same rational weight. If there exists a curve homotopic to a component γ_i of the simple lamination, then the orientation of γ_i agrees with one of the surfaces S or S° , and we assume that the curve lies entirely on this surface. By cutting along ∂S , we recover the surfaces S and S° with curves drawn on them.

Let \tilde{S} and \tilde{S}° denote the universal covers of S and S° , respectively. It is useful when drawing pictures to choose hyperbolic structures on S and S° . We can choose these hyperbolic structures so that if γ is a component of ∂S or ∂S° that does not meet a curve of the lamination, then the monodromy around γ is parabolic. Then the universal covers can be obtained from \mathbb{H} by removing countably many geodesic half disks. Below we will assume that these hyperbolic structures have been specified. None of our constructions will depend on the choice of hyperbolic structures.

Lemma 4.2. There exists a map $\tilde{\iota} : \tilde{S} \rightarrow \tilde{S}^\circ$, unique up to the action of $\pi_1(S)$ by deck transformations, such that the diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\iota}} & \tilde{S}^\circ \\ \downarrow & & \downarrow \\ S & \xrightarrow{\iota} & S^\circ \end{array}$$

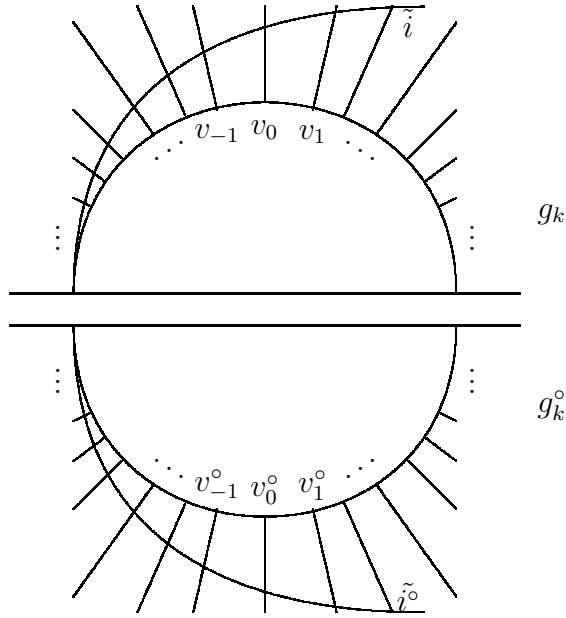
commutes. Points on $\partial\tilde{S}$ that project to points on the curves of the lamination are mapped bijectively to points on $\partial\tilde{S}^\circ$ that project to points on the curves, and this bijection preserves the natural order of these points.

Proof. The lifting criterion ensures that there is a map $\tilde{\iota} : \tilde{S} \rightarrow \tilde{S}^\circ$ lifting the composition $\tilde{S} \rightarrow S \rightarrow S^\circ$. It is easy to check that this map has the required properties. \square

4.2.2 Definition of b_i and x_i

To construct coordinates on $\mathcal{D}_L(S, \mathbb{Q})$, fix a point $l \in \mathcal{D}_L(S, \mathbb{Q})$ and let $i \in J$ be an edge of the ideal triangulation T . Realize the lamination l by a collection of nonintersecting simple closed curves on $S_{\mathcal{D}}$ such that all curves have the same rational weight. By cutting along the image of ∂S , we recover the surfaces S and S° with curves drawn on them.

Suppose the edge i corresponds to an arc connecting two holes γ_1 and γ_2 in S which meet the curves of l . Choose a pair of geodesics g_1 and g_2 in $\partial\tilde{S}$ which project to these boundary components. Deform the curve connecting the holes by winding its endpoint around the holes infinitely many times in the direction prescribed by the orientation. This corresponds to deforming the preimage \tilde{i} in \tilde{S} so that its endpoints coincide with endpoints of g_1 and g_2 . If we let i° denote the image of i under the tautological map $S \rightarrow S^\circ$, then we can apply the same procedure to i° to get an arc \tilde{i}° in \tilde{S}° .



Observe that the curves of the lamination that end on γ_k can be lifted to infinitely many curves in \tilde{S} that end on g_k . Similarly, if g_k° is the geodesic in \tilde{S}° that projects to γ_k , then the curves in S° that end on γ_k can be lifted to infinitely many curves in \tilde{S}° that end on g_k° . Label the endpoints of curves on g_k by the symbols v_α ($\alpha \in \mathbb{Z}$), and label the endpoints of curves on g_k° by v_α° ($\alpha \in \mathbb{Z}$). By Lemma 4.2, there is a map $\tilde{\iota} : \tilde{S} \rightarrow \tilde{S}^\circ$ that projects to the natural map $\iota : S \rightarrow S^\circ$ and is unique up to the action of $\pi_1(S)$ by deck transformations.

This provides a bijection

$$f : \{\text{vertices } v_\alpha\} \rightarrow \{\text{vertices } v_\alpha^\circ\}$$

which preserves the order of the vertices. Choose a vertex $v_{\alpha(k)}$ on g_k and let $v_{\beta(k)}^\circ$ be the corresponding vertex given by $v_{\beta(k)}^\circ = f(v_{\alpha(k)})$. We can choose $v_{\alpha(k)}$ in such a way that the curve ending at $v_{\alpha(k)}$ intersects \tilde{i} and the curve ending at $v_{\beta(k)}^\circ$ intersects \tilde{i}° . Notice that if we choose a different vertex $v_{\alpha(k)}$, then the vertex $v_{\beta(k)}^\circ$ will change by a corresponding amount.

By construction, the curve that ends at $v_{\alpha(k)}$ intersects \tilde{i} at some point $p(k)$. Denote by a_i half the number of intersections between the lifted curves of l and the lifted edge \tilde{i} between the points $p(1)$ and $p(2)$. Similarly, the curve that ends at $v_{\beta(k)}^\circ$ intersects \tilde{i}° at some point $p^\circ(k)$. Let a_i° be half the number of intersections between the lifted curves of l and the lifted edge \tilde{i}° between the points $p^\circ(1)$ and $p^\circ(2)$. We can then define a coordinate associated to the edge i by

$$b_i = a_i^\circ - a_i.$$

This defines b_i when i corresponds to an arc connecting two holes. If one or both of the endpoints of i are punctures, then i intersects only finitely many curves near these punctures, and i° intersects only finitely many curves near the corresponding punctures in S° . Thus we can associate the half intersection numbers a_i and a_i° to these arcs as before, and we can define b_i by the above formula. One can show that the $|J|$ numbers obtained in this way are independent of all choices made in the construction.

In addition to the b_i , there are $|J|$ numbers x_i associated to a point in the space $\mathcal{D}_L(S, \mathbb{Q})$. Given a point of $\mathcal{D}_L(S, \mathbb{Q})$, these are simply defined as the X -coordinates of the \mathcal{X} -lamination on the surface S obtained by cutting $S_{\mathcal{D}}$ along the image of ∂S .

4.2.3 Reconstruction

We will now show that the b_i and x_i are indeed coordinates on $\mathcal{D}^+(S)$.

Proposition 4.3. The numbers b_i and x_i provide a bijection

$$\mathcal{D}_L(S, \mathbb{Q}) \rightarrow \mathbb{Q}^{2|J|}.$$

Proof. Call this map ϕ . We will construct a map $\psi : \mathbb{Q}^{2|J|} \rightarrow \mathcal{D}_L(S, \mathbb{Q})$ and prove that ϕ and ψ are inverses.

Step 1. Definition of $\psi : \mathbb{Q}^{2|J|} \rightarrow \mathcal{D}_L(S, \mathbb{Q})$.

Suppose we are given rational numbers b_j and x_j for every $j \in J$. By the reconstruction procedure for \mathcal{X} -laminations, we can use the x_j to glue together ideal triangles to get a region $\tilde{S} \subseteq \mathbb{H}$ with geodesic boundary together with infinitely many curves.

Let t_0 be a triangle in \tilde{S} . There are infinitely many curves connecting each pair of adjacent sides of this triangle. For each vertex p of t_0 , choose a curve connecting the two sides of this triangle that meet at p . Then for every edge i , there is a number a_i defined as half the total weight of the curves that intersect i between the distinguished curves. Define

$$a_i^\circ = b_i + a_i.$$

We will use these numbers a_i° to construct another region \tilde{S}° with geodesic boundary in a copy of \mathbb{H} .

To construct \tilde{S}° , let u_0 be an ideal triangle with infinitely many curves connecting each pair of adjacent sides as before. If we choose the distinguished curves on t_0 sufficiently close to the vertices, then we can choose a triple of distinguished curves near the vertices of u_0 so that a_i° is half the total weight of the curves that intersect an edge of u_0 between distinguished curves.

Now suppose t is a triangle adjacent to t_0 in the triangulation of \tilde{S} . As before we can choose a distinguished curve near each vertex of t to get a triple of numbers a_i . Then we can draw an ideal triangle u with infinitely many curves connecting each pair of adjacent sides, and there are distinguished curves near each vertex of u realizing the numbers a_i° . We can choose the distinguished curves on t_0 and t so that they coincide at the common edge $t_0 \cap t$, and then we can glue u_0 and u so that their distinguished curves coincide. Continuing this process inductively, we obtain the desired space \tilde{S}° .

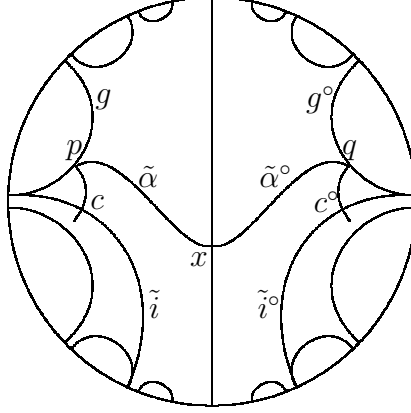
Now the region \tilde{S} is obtained from \mathbb{H} by removing infinitely many geodesic half disks. We can extend \tilde{S} to a larger region by gluing a copy of \tilde{S}° in each of these half disks in such a way that corresponding curves are identified. The resulting region is again obtained from \mathbb{H} by removing infinitely many geodesic half disks, and we can enlarge it by gluing a copy of \tilde{S} in each of these half disks. Continuing this process ad infinitum, we partition the hyperbolic plane into ideal triangles together with infinitely many curves. Quotienting this space by the group of deck transformations, we recover the surface $S_{\mathcal{D}}$ with a collection of curves. There may be infinitely many curves homotopic to some component γ_i of γ . In this case we delete a maximal collection of such curves between the distinguished curves. If there are any remaining curves homotopic to γ_i , then they all lie on one of the surfaces S or S° , and we choose the orientation of γ_i to agree with the orientation of this surface.

Step 2. Proof of $\phi \circ \psi = 1_{\mathbb{Q}^{2|J|}}$.

Let b_j and x_j ($j \in J$) be given. By the reconstruction procedure described above, we construct triangulated regions $\tilde{S} \subseteq \mathbb{H}$ and $\tilde{S}^\circ \subseteq H$ which we then glue together to get a triangulation of the hyperbolic plane together with a collection of curves. From this data, we get a \mathcal{D} -lamination l on $S_{\mathcal{D}}$. We want to show that the coordinates of l are the numbers b_j and x_j .

We can lift l to its universal cover, which is exactly the triangulated surface $\tilde{S}_{\mathcal{D}}$ that we got by gluing together copies of \tilde{S} and \tilde{S}° . Consider a copy of \tilde{S} in $\tilde{S}_{\mathcal{D}}$, and let \tilde{i} be an edge of the triangulation of \tilde{S} . To find the coordinate of l corresponding to the edge \tilde{i} , we must choose near each endpoint of \tilde{i} a distinguished curve in \tilde{S} which intersects \tilde{i} . Now consider a copy of \tilde{S}° adjacent to \tilde{S} in $\tilde{S}_{\mathcal{D}}$. Choose a basepoint x on the geodesic separating these regions. Suppose the edge \tilde{i} is asymptotic to an endpoint of the boundary geodesic g of \tilde{S} . Then there is a distinguished curve c that intersects \tilde{i} and g . Let p be the point of

intersection with g . Draw a curve $\tilde{\alpha}$ in \tilde{S} from the point p to the point x .



This curve $\tilde{\alpha}$ projects to a curve α on the surface S , and we can apply the map ι to get a curve α° on S° . Finally, we lift α° to a curve $\tilde{\alpha}^\circ$ in S° that starts at x . This curve $\tilde{\alpha}^\circ$ ends at some point $q = \tilde{\iota}(p)$, and there is a unique lifted curve c° of the lamination that passes through q . In this way, we get distinguished curves near the endpoints of the edge \tilde{i}° . These can be used to define the coordinates of l .

On the other hand, consider the monodromy along the curve in $S_{\mathcal{D}}$ obtained by concatenating α and α° . It maps the region \tilde{S} isometrically into the region bounded by g° . In the construction of l from the coordinates b_j and x_j , there is a correspondence between curves that end on g and curves that end on g° , and this correspondence is obtained by applying this transformation. Thus the distinguished curves used to construct l agree with the ones obtained using the map $\tilde{\iota}$.

It follows that the b -coordinates of l are simply the numbers b_j that we started with. It is easy to see that the x -coordinates of l are the numbers x_j . This completes Step 2 of the proof.

Step 3. Proof of $\psi \circ \phi = 1_{\mathcal{D}_L}$.

Let $l \in \mathcal{D}_L(S, \mathbb{Q})$ be given. To calculate the coordinates of l , let us pass to the universal cover $\tilde{S}_{\mathcal{D}}$ of $S_{\mathcal{D}}$ and choose a copy of \tilde{S} in $\tilde{S}_{\mathcal{D}}$. We associate numbers x_i to the edges of the triangulation in the usual way. If \tilde{i} is an edge of the triangulation of \tilde{S} , then we can choose a distinguished curve near each endpoint of \tilde{i} . Using the construction described above involving lifts of the map $\iota : S \rightarrow S^\circ$, we can get a pair of distinguished curves near the endpoints of a corresponding edge \tilde{i}° in \tilde{S}° . We can use these curves to calculate the numbers a_i and a_i° , and thus the coordinate b_i .

We can now use the x_i to reconstruct the region \tilde{S} . We must choose a distinguished curve near each endpoint of the edge \tilde{i} , and we can assume that these are exactly the ones used above to define b_i . We can use the curves to begin constructing the space \tilde{S}° with a collection of curves. By gluing together copies of \tilde{S} and \tilde{S}° , we recover $\tilde{S}_{\mathcal{D}}$ with a collection of curves. Quotienting this space by the action of $\pi_1(S_{\mathcal{D}})$, we recover the point $l \in \mathcal{D}_L(S, \mathbb{Q})$. This completes Step 3 of the proof. \square

4.2.4 Transformation rule

We have defined the b_i and x_i coordinates in terms of a fixed ideal triangulation of S . The next result says how these coordinates vary when we change the triangulation.

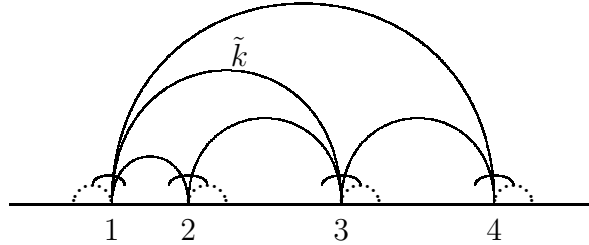
Proposition 4.4. A regular flip at an edge k of the triangulation changes the coordinates x_i and b_i to new coordinates x'_i and b'_i given by the formulas

$$x'_i = \begin{cases} -x_k & \text{if } i = k \\ x_i + \varepsilon_{ki} \max(0, \operatorname{sgn}(\varepsilon_{ki})x_k) & \text{if } i \neq k \end{cases}$$

$$b'_i = \begin{cases} \max\left(x_k + \sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj}b_j, -\sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj}b_j\right) - \max(0, x_k) - b_k & \text{if } i = k \\ b_i & \text{if } i \neq k. \end{cases}$$

Proof. By definition, the coordinates x_i on $\mathcal{D}_L(S, \mathbb{Q})$ coincide with the coordinates of the \mathcal{X} -lamination on the surface S . Therefore the transformation rule for the x_i is just the usual transformation rule for the coordinates of an \mathcal{X} -lamination.

To prove the second rule, deform the curves corresponding to edges of the ideal triangulation by winding their endpoints around the holes infinitely many times. Lift these deformed curves to the universal cover of S , and consider the ideal quadrilateral formed by the two triangles adjacent to the geodesic arc \tilde{k} corresponding to k .



Number the vertices of this quadrilateral in counterclockwise order so that the edge \tilde{k} joins vertices 1 and 3. In the construction of the b_i , we considered near each endpoint e of a lifted edge \tilde{i} a curve that intersects \tilde{i} and projects down to a curve of the lamination. We can choose these curves to intersect both edges of the quadrilateral that meet at e .

Our construction will then associate a number a_{ij} to the geodesic connecting i and j . There is a corresponding ideal quadrilateral in the universal cover of S° and a number a_{ij}° corresponding to the edge connecting vertices i and j of this ideal quadrilateral.

If we flip at the edge k , then the ideal quadrilateral in the universal cover of S is replaced by the same ideal quadrilateral triangulated by the arc from 2 to 4, and the number associated to this new arc is

$$a_{24} = \max(a_{12} + a_{34}, a_{14} + a_{23}) - a_{13}.$$

Similarly, the number a_{13}° transforms to

$$a_{24}^\circ = \max(a_{12}^\circ + a_{34}^\circ, a_{14}^\circ + a_{23}^\circ) - a_{13}^\circ.$$

Applying these rules to the difference $b_{13} = a_{13}^\circ - a_{13}$, we obtain

$$\begin{aligned}
b_{24} &= \max(a_{12}^\circ + a_{34}^\circ, a_{14}^\circ + a_{23}^\circ) - \max(a_{12} + a_{34}, a_{14} + a_{23}) - (a_{13}^\circ - a_{13}) \\
&= \max(a_{12}^\circ + a_{34}^\circ - a_{14} - a_{23}, a_{14}^\circ + a_{23}^\circ - a_{12} - a_{23}) \\
&\quad - \max(0, a_{12} + a_{34} - a_{14} - a_{23}) - (a_{13}^\circ - a_{13}) \\
&= \max(x_{13} + b_{12} + b_{34}, b_{14} + b_{23}) - \max(0, x_{13}) - b_{13}
\end{aligned}$$

where we have used the relation $x_{13} = a_{12} + a_{34} - a_{14} - a_{23}$. This proves the transformation rule for the b_i . \square

4.3 Real \mathcal{D} -laminations

Since the transformation rules in Proposition 4.4 are continuous with respect to the standard topology on $\mathbb{Q}^{2|J|}$, the coordinates define a natural topology on $\mathcal{D}_L(S, \mathbb{Q})$. We define the space of *real \mathcal{D} -laminations* as the metric space completion of $\mathcal{D}_L(S, \mathbb{Q})$.

This space of real \mathcal{D} -laminations is identified with the space $\mathcal{D}(\mathbb{R}^t)$ of \mathbb{R}^t -points of the symplectic double associated to the surface S . There is a natural action of the group $\mathbb{R}_{>0}$ on this space where an element $\lambda \in \mathbb{R}_{>0}$ acts by multiplying the coordinates in any coordinate system by λ . The *spherical tropical space* $\mathcal{SD}(\mathbb{R}^t)$ is the quotient

$$\mathcal{SD}(\mathbb{R}^t) = (\mathcal{D}(\mathbb{R}^t) - 0) / \mathbb{R}_{>0}.$$

By Proposition 2.2 of [FG5], we know that $\mathcal{SD}(\mathbb{R}^t)$ can be viewed as a boundary of the space $\mathcal{D}^+(S)$ of positive real points of the symplectic double associated to the surface S .

5 Cluster algebras and F -polynomials

5.1 Cluster algebras with coefficients

In the following definition, \mathbb{P} denotes an arbitrary semifield. One can show that the group ring $\mathbb{Z}\mathbb{P}$ is an integral domain, and hence we can form its fraction field $\mathbb{Q}\mathbb{P}$. We will write \mathcal{F} for a field isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{Q}\mathbb{P}$.

Definition 5.1. A *labeled seed* $(\mathbf{x}, \mathbf{y}, B)$ consists of a skew-symmetrizable $n \times n$ integer matrix $B = (b_{ij})$, an n -tuple $\mathbf{y} = (y_1, \dots, y_n)$ of elements of \mathbb{P} , and an n -tuple $\mathbf{x} = (x_1, \dots, x_n)$ of elements of \mathcal{F} such that the x_i are algebraically independent over $\mathbb{Q}\mathbb{P}$ and $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \dots, x_n)$.

Definition 5.2. Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed, and let $k \in \{1, \dots, n\}$. Then we define a new seed $(\mathbf{x}', \mathbf{y}', B')$, called the seed obtained by *mutation* in the direction k as follows:

1. The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{if } k \notin \{i, j\}. \end{cases}$$

2. The elements of the n -tuple $\mathbf{y}' = (y'_1, \dots, y'_n)$ are given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \end{cases}$$

where we are using the notation $[b]_+ = \max(b, 0)$.

3. The elements of the n -tuple $\mathbf{x}' = (x'_1, \dots, x'_n)$ are given by

$$x'_j = \begin{cases} \frac{y_k \prod_{i|b_{ik}>0} x_i^{b_{ik}} + \prod_{i|b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1)x_k} & \text{if } j = k \\ x_j & \text{if } j \neq k. \end{cases}$$

Definition 5.3. We denote by \mathbb{T}_n an n -regular tree with edges labeled by the numbers $1, \dots, n$ in such a way that the n edges emanating from any vertex have distinct labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$ so that if t and t' are vertices connected by an edge labeled k , then $\Sigma_{t'}$ is obtained from Σ_t by a mutation in the direction k . We will use the following notation for the elements of Σ_t :

$$\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \dots, y_{n;t}), \quad B_t = (b_{ij}^t).$$

Definition 5.4. Given a cluster pattern $t \mapsto (\mathbf{x}_t, \mathbf{y}_t, B_t)$, we form the set of all cluster variables in all seeds of the cluster pattern:

$$\mathcal{S} = \{x_{l;t} : t \in \mathbb{T}_n, 1 \leq l \leq n\}.$$

Then the *cluster algebra* with coefficients in \mathbb{P} is the \mathbb{ZP} -subalgebra of \mathcal{F} generated by elements in this set \mathcal{S} .

5.2 Principal coefficients

In [FZIV], Fomin and Zelevinsky define a *cluster algebra with principal coefficients* at a vertex $t_0 \in \mathbb{T}_n$ to be a cluster algebra with $\mathbb{P} = \text{Trop}(y_1, \dots, y_n)$ and $\mathbf{y}_{t_0} = (y_1, \dots, y_n)$. Let \mathcal{A} be such a cluster algebra, and let $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be its initial seed with

$$\mathbf{x}_{t_0} = (x_1, \dots, x_n), \quad \mathbf{y}_{t_0} = (y_1, \dots, y_n), \quad B_{t_0} = (b_{ij}^0).$$

By iterating the exchange relations, we can express any cluster variable $x_{l;t}$ as a subtraction-free rational function of the variables $x_1, \dots, x_n, y_1, \dots, y_n$. We will denote this subtraction-free rational function by

$$X_{l;t} \in \mathbb{Q}_{\text{sf}}(x_1, \dots, x_n, y_1, \dots, y_n).$$

We will denote by $F_{l;t} \in \mathbb{Q}_{\text{sf}}(y_1, \dots, y_n)$ the subtraction-free rational function obtained from $X_{l;t}$ by specializing all the x_i to 1. Thus

$$F_{l;t}(y_1, \dots, y_n) = X_{l;t}(1, \dots, 1, y_1, \dots, y_n).$$

By the Laurent phenomenon theorem of Fomin and Zelevinsky, $X_{l;t}$ is a Laurent polynomial in x_1, \dots, x_n whose coefficients are integral polynomials in y_1, \dots, y_n , and $F_{l;t}$ is an integral polynomial in y_1, \dots, y_n . We will refer to $X_{l;t}$ and $F_{l;t}$ as the X - and F -polynomials.

In addition to the X - and F -polynomials, Fomin and Zelevinsky define a \mathbb{Z}^n -grading on the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_n]$ by

$$\deg(x_i) = \mathbf{e}_i, \quad \deg(y_j) = -\mathbf{b}_j^0$$

where \mathbf{e}_i is the i th standard basis vector in \mathbb{Z}^n and $\mathbf{b}_j^0 = \sum_i b_{ij}^0 \mathbf{e}_i$ is the j th column of B_{t_0} . By a result of [FZIV], each X -polynomial is homogeneous with respect to this \mathbb{Z}^n -grading. The degree

$$\mathbf{g}_{l;t} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \deg(X_{l;t}) \in \mathbb{Z}^n$$

is called the \mathbf{g} -vector of the cluster variable $x_{l;t}$.

The notions of F -polynomials and \mathbf{g} -vectors are important because they allow us to express an arbitrary cluster variable in terms of the variables x_1, \dots, x_n and y_1, \dots, y_n of the initial seed Σ_{t_0} . To see this, we need one more piece of notation. It is a fact that any subtraction-free rational identity that holds in the semifield $\mathbb{Q}_{\text{sf}}(u_1, \dots, u_n)$ will remain valid when we replace the u_i by elements of an arbitrary semifield \mathbb{P} . Thus if f is a subtraction-free rational expression in u_1, \dots, u_n , there is a well defined element $f|_{\mathbb{P}}(y_1, \dots, y_n)$ of \mathbb{P} obtained by evaluating f at $y_1, \dots, y_n \in \mathbb{P}$.

Proposition 5.5 ([FZIV], Corollary 6.3). Let \mathcal{A} be a cluster algebra over an arbitrary semifield \mathbb{P} of coefficients. Then a cluster variable $x_{l;t}$ can be expressed in terms of the cluster variables at an initial seed as

$$x_{l;t} = \frac{F_{l;t}|_{\mathcal{F}}(\widehat{y}_1, \dots, \widehat{y}_n)}{F_{l;t}|_{\mathbb{P}}(y_1, \dots, y_n)} x_1^{g_1} \dots x_n^{g_n}$$

where

$$\widehat{y}_j = y_j \prod_i x_i^{b_{ij}^0}.$$

5.3 Relation to the symplectic double

Let us now specialize to the case where $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$ is the semifield of subtraction-free rational functions in X_1, \dots, X_n .

Proposition 5.6. If $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$, then the ambient field \mathcal{F} is given by $\mathcal{F} = \mathbb{Q}(X_1, \dots, X_n, B_1, \dots, B_n)$ for algebraically independent variables B_1, \dots, B_n .

Proof. If $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$, then we have $\mathbb{Q}\mathbb{P} = \mathbb{Q}(X_1, \dots, X_n)$ and hence

$$\begin{aligned} \mathcal{F} &= \mathbb{Q}(X_1, \dots, X_n)(B_1, \dots, B_n) \\ &= \mathbb{Q}(X_1, \dots, X_n, B_1, \dots, B_n) \end{aligned}$$

as desired. □

Proposition 5.7. If $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$, then a mutation in the direction k transforms the n -tuples (X_1, \dots, X_n) and (B_1, \dots, B_n) into n -tuples (X'_1, \dots, X'_n) and (B'_1, \dots, B'_n) given by

$$X'_j = \begin{cases} X_k^{-1} & \text{if } j = k \\ X_j(1 + X_k^{-\text{sgn}(\varepsilon_{jk})})^{-\varepsilon_{jk}} & \text{if } j \neq k \end{cases}$$

and

$$B'_j = \begin{cases} \frac{X_k \prod_{i|\varepsilon_{ki}>0} B_i^{\varepsilon_{ki}} + \prod_{i|\varepsilon_{ki}<0} B_i^{-\varepsilon_{ki}}}{B_k(1+X_k)} & \text{if } i = k \\ B_j & \text{if } i \neq k \end{cases}$$

where $\varepsilon_{ij} = b_{ji}$.

Proof. The second formula obviously follows from the general mutation formula. To prove the first formula, observe that

$$\begin{aligned} X_j X_k^{[b_{kj}]_+} (X_k \oplus 1)^{-b_{kj}} &= X_j X_k^{[b_{kj}]_+} (X_k + 1)^{-b_{kj}} \\ &= X_j X_k^{-(-b_{kj})} (X_k + 1)^{-b_{kj}} \\ &= X_j (X_k^{-1} X_k + X_k^{-1})^{-b_{kj}} \\ &= X_j (1 + X_k^{-\text{sgn}(b_{kj})})^{-b_{kj}} \end{aligned}$$

for $b_{kj} > 0$ and

$$\begin{aligned} X_j X_k^{[b_{kj}]_+} (X_k \oplus 1)^{-b_{kj}} &= X_j (1 + X_k)^{-b_{kj}} \\ &= X_j (1 + X_k^{-\text{sgn}(b_{kj})})^{-b_{kj}} \end{aligned}$$

for $b_{kj} \leq 0$. □

Thus we recover the mutation formulas for the symplectic double in the special case where $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$. In this case, we will denote a cluster algebra with coefficients in \mathbb{P} by the symbol \mathcal{D} and call it a *cluster \mathcal{D} -algebra*.

5.4 Cluster algebras from surfaces

5.4.1 Construction of cluster algebras

In [FST], Fomin, Shapiro, and Thurston discuss the relationship between cluster algebras and the combinatorics of decorated surfaces. Here we will review their work and apply it to the cluster \mathcal{D} -algebra that we defined above.

The idea of [FST] is to associate to a decorated surface S a corresponding cluster algebra. This cluster algebra is defined in such a way that each seed corresponds to a “tagged

triangulation” of the surface S . An ordinary ideal triangulation is a special case of a tagged triangulation provided there are no self-folded triangles. Fomin, Shapiro, and Thurston assume that the surface S is not a sphere with one, two, or three punctures, a monogon with zero or one puncture, or a bigon or triangle without punctures. According to Lemma 2.13 of [FST], such a surface always admits an ideal triangulation T with no self-folded triangles.

If T is an ideal triangulation of S with no self-folded triangles, then we get an exchange matrix $b_{ij} = \varepsilon_{ji}$ ($i, j \in J$), indexed by the internal edges of T . To each internal edge i of T , we associate variables x_i and y_i . This defines a labeled seed, and hence a cluster algebra. This is the cluster algebra that Fomin, Shapiro, and Thurston associate to the surface S .

If c is any arc on S which is an internal edge for some ideal triangulation and does not cut out a once punctured monogon, then there is a cluster variable x_c in this cluster algebra corresponding to c . In particular, this means that for any such arc c on S there is an associated F -polynomial

$$F_c(y_1, \dots, y_n)$$

and a \mathbf{g} -vector \mathbf{g}_c .

We can apply the results of [FST] in the special case where the semifield of coefficients is a semifield of subtraction-free rational functions $\mathbb{P} = \mathbb{Q}_{\text{sf}}(X_1, \dots, X_n)$. In this way, we get a cluster \mathcal{D} -algebra associated to a surface S . Applying Proposition 5.5 to this cluster algebra, we obtain the following result.

Proposition 5.8. Let \mathcal{D} be the cluster \mathcal{D} -algebra associated to a surface S , and let B_c be a cluster variable of \mathcal{D} corresponding to an arc c on S . Then B_c can be expressed in terms of the initial variables B_1, \dots, B_n and X_1, \dots, X_n by the formula

$$B_c = \frac{F_c(\hat{X}_1, \dots, \hat{X}_n)}{F_c(X_1, \dots, X_n)} B_1^{g_1} \dots B_n^{g_n}$$

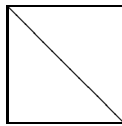
where

$$\hat{X}_i = X_i \prod_j B_j^{\varepsilon_{ij}}.$$

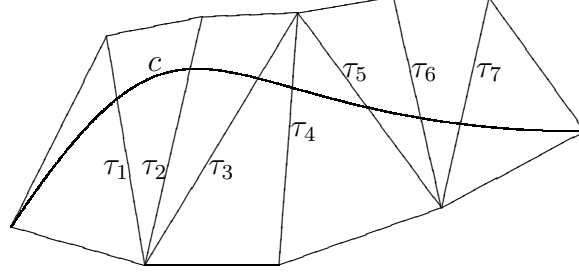
In Section 6, we will discuss an extension of this result which will allow us to completely understand the canonical pairing between \mathcal{D} -laminations and points of the Teichmüller \mathcal{D} -space. In order to prove this result, we will need to understand the \mathbf{g} -vectors associated to arcs in S .

5.4.2 Calculation of \mathbf{g} -vectors

As part of their work on the positivity conjecture for cluster algebras from surfaces, Musiker, Schiffler, and Williams [MSW1] gave a formula for computing the \mathbf{g} -vector associated to an arc. Their construction associates, to any arc c , a graph $\bar{G}_{T,c}$ in the plane with labeled edges. This graph is obtained by gluing together “tiles” of the form



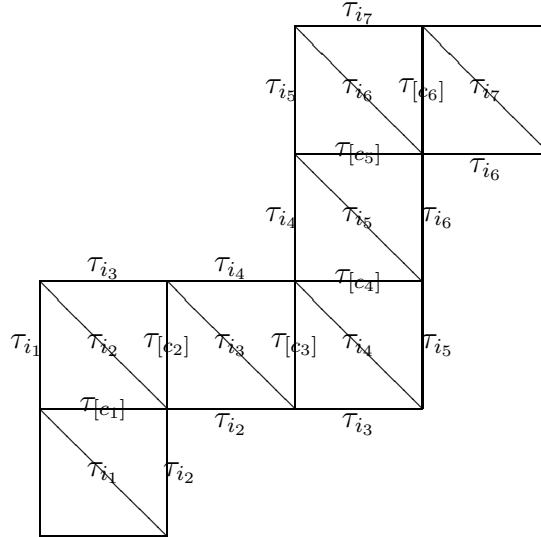
Indeed, suppose c is an arc on a triangulated unpunctured surface. (We refer the reader to [MSW1] for the case of a surface with punctures, which is similar.) Assume that this arc is not an edge of the triangulation. The illustration below shows an example of such a curve on a disk with ten marked points.



Choose an orientation for c , and label the arcs that c crosses in order by $\tau_{i_1}, \dots, \tau_{i_d}$. For any index j , let Δ_{j-1} and Δ_j be the two triangles on either side of τ_{i_j} . Then we can associate a tile G_j as above to each τ_{i_j} . It consists of two triangles with edges labeled as in Δ_{j-1} and Δ_j and glued together along the edge labeled τ_{i_j} so that the orientations of these triangles both agree or both disagree with those of Δ_{j-1} and Δ_j . Note that there are two possible planar embeddings of the graph G_j .

The two arcs τ_{i_j} and $\tau_{i_{j+1}}$ are edges of the triangle Δ_j . We will write $\tau_{[c_j]}$ for the third arc in this triangle. Then we can recursively glue together the tiles in order from 1 to d so that G_{j+1} and G_j are glued along the edges labeled $\tau_{[c_j]}$ and if the orientation of the triangles of G_j agrees with the orientation of Δ_{j-1} and Δ_j then the orientation of G_{j+1} disagrees with the orientation of Δ_j and Δ_{j+1} , and vice versa. We denote the resulting graph by $\bar{G}_{T,c}$.

For example, the graph $\bar{G}_{T,c}$ corresponding to the above example is



Write $G_{T,c}$ for the graph obtained from $\bar{G}_{T,c}$ by removing the diagonal in every tile. Recall that for any graph G , a *perfect matching* of G is a collection P of edges such that every vertex of G is incident to exactly one edge in P . It is easy to show that the graph $G_{T,c}$

constructed in [MSW1] has exactly two perfect matchings consisting only of boundary edges. These perfect matchings are called the *minimal matching* and *maximal matching* and are denoted $P_- = P_-(G_{T,c})$ and $P_+ = P_+(G_{T,c})$, respectively. In the above example, the maximal matching P_+ is the matching that contains the horizontal edge at the bottom of the graph $\tilde{G}_{T,c}$.

If the edges of a perfect matching P are labeled $\tau_{j_1}, \dots, \tau_{j_r}$, then we define the *weight* $x(P)$ of P as the product

$$x(P) = \prod_{s=1}^r x_{\tau_{j_s}}$$

of the cluster variables associated to $\tau_{j_1}, \dots, \tau_{j_r}$. Similarly, if $\tau_{i_1}, \dots, \tau_{i_d}$ is the sequence of arcs in T that c crosses, then we define the *crossing monomial* $\text{cross}(T, c)$ of c with respect to T as the product

$$\text{cross}(T, c) = \prod_{s=1}^d x_{\tau_{i_s}}.$$

Note that the arcs $\tau_{i_1}, \dots, \tau_{i_d}$ in this definition also appear as the labels on the diagonal edges in the graph $\tilde{G}_{T,c}$.

Proposition 5.9 ([MSW1]). Let c be an arc on a decorated surface S . Then the \mathbf{g} -vector associated to c is given by the formula

$$\mathbf{g}_c = \deg \left(\frac{x(P_-)}{\text{cross}(T, c)} \right).$$

6 The canonical pairing

6.1 The multiplicative canonical pairing

A simple closed curve l on $S_{\mathcal{D}}$ will be called an *intersecting curve* if every curve in its homotopy class intersects the image of ∂S in $S_{\mathcal{D}}$.

Suppose we are given an intersecting curve l on $S_{\mathcal{D}}$. Deform this curve so that it intersects the image of ∂S in the minimal number of points. Then, starting from any component of this image which we may call γ_1 , there is a segment c_1 of the curve l which lies entirely in S and connects the component γ_1 to another component which we may call γ_2 . Starting from this component, there is a segment c_2 of l which lies entirely in S° and connects γ_2 to a component γ_3 . Continue labeling in this way until the curve closes.

Now suppose we are given a point $m \in \mathcal{D}^+(S)$. This determines a point in the Teichmüller space of $S_{\mathcal{D}}$ which we can represent by a hyperbolic structure on $S_{\mathcal{D}}$ such that $\partial S \subseteq S_{\mathcal{D}}$ is geodesic. Cut the surface $S_{\mathcal{D}}$ along ∂S . Then each c_i is a curve that connects two boundary components of S or S° . We can wind the ends of c_i around the holes infinitely many times in the direction prescribed by the orientations from m .

Consider a curve c_i for i odd. Lifting this curve to the upper half plane, we obtain a geodesic connecting two points on the boundary of \mathbb{H} . Choose a horocycle around each of

these boundary points, and define A_{c_i} as the exponentiated signed half length of the portion of the lifted curve between these horocycles. Next consider c_i for i even. Lifting to the upper half plane, we again get a geodesic connecting two points on the boundary of \mathbb{H} , and the horocycles already chosen determine a pair of horocycles around these points. We define $A_{c_i}^\circ$ as the exponentiated signed half length of the portion of the lifted curve between these horocycles.

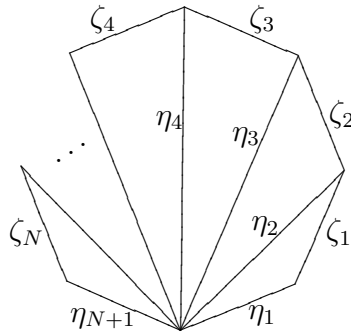
Definition 6.1. Fix a point $m \in \mathcal{D}^+(S)$.

1. Let l be an intersecting curve of weight k on $S_{\mathcal{D}}$, and assume that the orientation of each component of γ agrees with the orientation of S . Then

$$\mathbb{I}_{\mathcal{D}}(l, m) = \left(\frac{\prod_{i \text{ even}} A_{c_i}^\circ}{\prod_{i \text{ odd}} A_{c_i}} \right)^k.$$

2. Let l be a curve of weight k on $S_{\mathcal{D}}$ which is not an intersecting curve and is not homotopic to a loop in the simple lamination. If l lies in S° , then $\mathbb{I}_{\mathcal{D}}(l, m)$ is defined as the absolute value of the trace of the k th power of the of the monodromy around l . If l lies in S then $\mathbb{I}_{\mathcal{D}}(l, m)$ is the reciprocal of this quantity.
3. Let l be a curve of weight k on $S_{\mathcal{D}}$ which is homotopic to a loop γ_i in the simple lamination. If l and m provide the same orientation for γ_i , then $\mathbb{I}_{\mathcal{D}}(l, m)$ is defined as the absolute value of the k th power of the largest eigenvalue of the monodromy around l . If l and m provide different orientations for γ_i , then $\mathbb{I}_{\mathcal{D}}(l, m)$ is the reciprocal of this quantity.
4. Let l_1 and l_2 be laminations on $S_{\mathcal{D}}$ such that no curve from l_1 intersects or is homotopic to a curve from l_2 . Then $\mathbb{I}_{\mathcal{D}}(l_1 + l_2, m) = \mathbb{I}_{\mathcal{D}}(l_1, m)\mathbb{I}_{\mathcal{D}}(l_2, m)$.

This defines the canonical map in the special case where the orientation of any γ_i that meets a curve agrees with the orientation of the surface S . If the orientation of γ_i disagrees with the orientation of S , then we can define $\mathbb{I}_{\mathcal{D}}(l)$ by modifying slightly the above definition. To do this, we first note that there is a natural invariant associated to each γ_i that meets a curve of l . Indeed, fix an ideal triangulation T of S , and lift this to an ideal triangulation \tilde{T} of the universal cover \tilde{S} . Consider the triangles t_1, \dots, t_N of T that meet the puncture in S corresponding to γ_i . We can find an ideal polygon in \tilde{S} , formed by N triangles of \tilde{T} , which projects onto the union of the triangles t_1, \dots, t_N . We will label the edges of this ideal polygon as in the diagram below.



Choose a horocycle at each vertex of this polygon. There is a similar polygon in the universal cover \tilde{S}° of S° , and the horocycles already chosen determine a corresponding collection of horocycles associated to the vertices of this polygon. We denote the corresponding edges of the triangulation of this new polygon by the same symbols $\eta_1, \dots, \eta_{N+1}$ and ζ_1, \dots, ζ_N , and we write A_j° for the invariant associated to the edge j . We can then form the expression

$$\kappa_i = \frac{\alpha^\circ}{\alpha}$$

where

$$\alpha^\circ = \sum_{j=1}^N \frac{A_{\zeta_j}^\circ}{A_{\eta_j}^\circ A_{\eta_{j+1}}^\circ}$$

and

$$\alpha = \sum_{j=1}^N \frac{A_{\zeta_j}}{A_{\eta_j} A_{\eta_{j+1}}}.$$

Definition 6.2. Suppose that l is a point in $\mathcal{D}_L(S, \mathbb{Z})$ for which the function $\mathbb{I}_{\mathcal{D}}(l)$ has been defined. Suppose the orientation that l provides for γ_i agrees with the orientation of S . If l' is the lamination obtained from l by reversing the orientation of γ_i , then we define $\mathbb{I}_{\mathcal{D}}(l')$ as $\mathbb{I}_{\mathcal{D}}(l)$ multiplied by a factor of κ_i for each curve of weight 1 of l that meets γ_i .

We will see below that this definition is independent of all the choices made in the construction and that κ_i can be written as a rational expression in the B_j and X_j for any choice of triangulation.

6.2 Expression in terms of F -polynomials

6.2.1 Intersecting curves

We will prove our formula for $\mathbb{I}_{\mathcal{D}}(l)$ in several steps. We begin by examining the special case where l is an intersecting curve. In this case, we will find that the formula involves the F -polynomials of Fomin and Zelevinsky.

Fix an ideal triangulation T of S , and a point $m \in \mathcal{D}^+(S)$. Then the universal cover of S can be identified with a subset of the hyperbolic plane \mathbb{H} , and the triangulation T can be lifted to a triangulation \tilde{T} of the universal cover. Using the natural map $S^\circ \rightarrow S$, we can draw all of the curves c_i on the surface S . We can then lift each curve to a geodesic \tilde{c}_i in the universal cover in such a way that \tilde{c}_i and \tilde{c}_{i+1} share a common endpoint. Let P be a triangulated ideal polygon, formed from triangles in \tilde{T} , which includes all of the triangles that the curves \tilde{c}_i pass through. Let $T(P)$ be the triangulation of P provided by the triangulation \tilde{T} . Choose a horocycle around each vertex of $T(P)$. Then we can define A_i as the exponentiated signed half length of an edge i of an ideal triangle between the chosen horocycles.

In exactly the same way, the universal cover of S° can be identified with a subset of the hyperbolic plane, and the triangulation T provides a triangulation \tilde{T}° of this universal cover.

The polygon P gives rise to a polygon P° in this universal cover. The latter polygon has a triangulation $T(P^\circ)$ provided by the triangulation \tilde{T}° . The horocycles that we already chose around the vertices of $T(P)$ provide horocycles around the vertices of $T(P^\circ)$. We can define A_i° as the exponentiated signed half length of any edge i of an ideal triangle between these horocycles.

It is convenient at this point to adopt a kind of multi-index notation. If $\mathbf{v} = (v_i)$ is a vector indexed by the edges of the triangulation $T(P)$, then we will write

$$A^{\mathbf{v}} = \prod_i A_i^{v_i}.$$

Similarly, if $\mathbf{v} = (v_i)$ is indexed by the edges of $T(P^\circ)$, then we will write

$$(A^\circ)^{\mathbf{v}} = \prod_i (A_i^\circ)^{v_i}.$$

Since there is a natural bijection between the edges of $T(P^\circ)$ and the edges of $T(P)$, the indexing sets for these vectors can be identified.

Lemma 6.3. For each i , there is an F -polynomial F_{c_i} and an integral vector \mathbf{g}_{c_i} , indexed by the edges of the triangulation $T(P)$, such that

$$A_{c_i} = F_{c_i}(X_1, \dots, X_n) A^{\mathbf{g}_{c_i}}$$

and

$$A_{c_i}^\circ = F_{c_i}(\widehat{X}_1, \dots, \widehat{X}_n) (A^\circ)^{\mathbf{g}_{c_i}}.$$

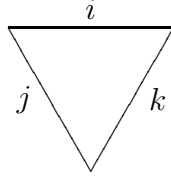
Proof. Let i_1, \dots, i_m be the edges of $T(P)$. Associated to the polygon P , there is a cluster algebra generated by the variables A_i over the trivial semifield $\mathbb{P} = \{1\}$. Applying Proposition 5.5 to this cluster algebra, we see that

$$A_{c_i} = F_{c_i} \left(\prod_j A_j^{\varepsilon_{i_1 j}}, \dots, \prod_j A_j^{\varepsilon_{i_m j}} \right) A_{i_1}^{g_{i_1}} \dots A_{i_m}^{g_{i_m}}$$

for some integers g_{i_1}, \dots, g_{i_m} . Now for any internal edge k of $T(P)$, the product $\prod_j A_j^{\varepsilon_{kj}}$ equals the coordinate X_k associated to the corresponding edge k of T . Moreover, by the matrix formula of [MW], this F_{c_i} is a polynomial only in the variables associated to edges that \tilde{c}_i crosses, which are all internal edges. Therefore we can write it as $F_{c_i}(X_1, \dots, X_n)$, a polynomial in the X -coordinates. This proves the first equation. The proof of the second equation is similar. In this case, one uses the fact that $\widehat{X}_k = \prod_j (A_j^\circ)^{\varepsilon_{kj}}$. \square

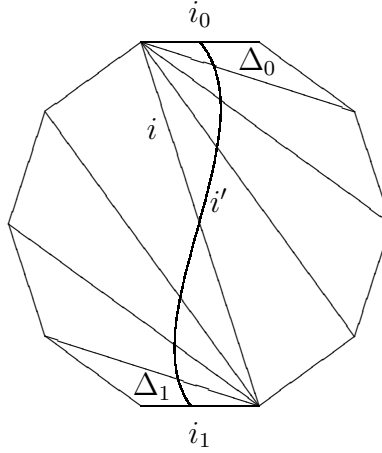
Recall that the variables X_k are defined as cross ratios $X_k = \frac{A_i A_m}{A_j A_l}$ where i, j, l, m are the edges of the quadrilateral with diagonal k . Here we will consider additional variables

associated to the edges i of \tilde{T} . Consider a triangle Δ in \tilde{T} that includes i as one of its edges, and label the other edges of this triangle as follows:



Then we define the variable associated to i by $W_{i,\Delta} = \frac{A_i A_j}{A_k}$. Note that this depends on the chosen triangle as well as the edge i .

Fix an edge i of the triangulation $T(P)$. For any vertex v of i , there is a collection of edges in \tilde{T} that start at v and lie in the counterclockwise direction from i . Consider a curve i' that goes diagonally across i , intersecting finitely many of these edges transversely before terminating on one of them. An example is illustrated below.



Given such a curve i' , let E_i be the set of all edges in $T(P)$ that i' crosses. Then we can form the product

$$P_i = W_{i_0, \Delta_0} \cdot \prod_{j \in E_i} X_j \cdot W_{i_1, \Delta_1}.$$

where i_0 and i_1 are the edges on which i' terminates. One can check that this expression equals A_i^2 . We will use this fact to prove the following result.

Lemma 6.4. Let $\mathbf{s} = \sum_{i \text{ even}} \mathbf{g}_{c_i} - \sum_{i \text{ odd}} \mathbf{g}_{c_i}$. Then there exists a half integral vector $\mathbf{h} = (h_i)_{i \in J}$, indexed by the internal edges of the triangulation T , such that

$$A^{\mathbf{s}} = X^{\mathbf{h}} = \prod_{i \in J} X_i^{h_i}.$$

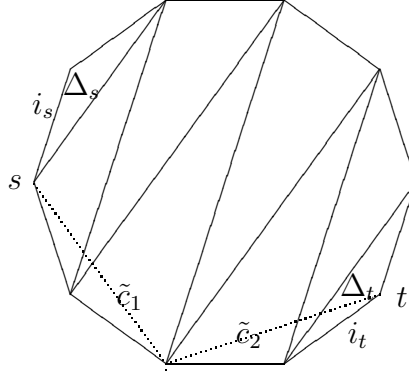
Proof. Consider an arc \tilde{c}_i in $T(P)$. If this arc coincides with an edge i of the triangulation, then the associated \mathbf{g} -vector equals the standard basis vector \mathbf{e}_i . Otherwise \tilde{c}_i intersects one

or more edges of $T(P)$, and we can compute the corresponding \mathbf{g} -vector \mathbf{g}_{c_i} using the results of [MSW1] that we reviewed above. In this case, \mathbf{g}_{c_i} is given by the formula

$$\mathbf{g}_{c_i} = \deg \left(\frac{x(P_-)}{\text{cross}(T, c_i)} \right).$$

By definition of a perfect matching, we know that any endpoint of a diagonal in the graph \bar{G}_{T, c_i} meets exactly one edge of the minimal perfect matching P_- . It follows that the vector \mathbf{g}_{c_i} is an alternating sum of standard basis vectors corresponding to the edges of a path in the graph \bar{G}_{T, c_i} .

Consider the path formed by the \tilde{c}_i in P . An example of such a path is illustrated below.



Let s and t be the endpoints of the path formed by the \tilde{c}_i . Consider the closed path on the surface S obtained by drawing all the c_i on S using the natural map $S^\circ \rightarrow S$. The horocycle at t is obtained from the one at s by applying the monodromy around this path. Let Δ_t be the triangle in P that contains t and is that last triangle that the lifted arcs c_i pass through. Let Δ_s be the preimage of this triangle under the monodromy.

By the above discussion, the vector \mathbf{s} is an alternating sum of standard basis vectors associated to the edges of a path in $T(P)$. To each edge i on this path, we associate a curve i' as above so that i' and j' terminate on a common edge whenever i and j terminate on a common vertex. We can choose these curves so that the first and last ones terminate at corresponding edges i_s of Δ_s and i_t of Δ_t . One can show in this case that $W_{i_s, \Delta_s} = W_{i_t, \Delta_t}$. It follows from the above discussion that

$$\prod_i P_i^{s_i} = \left(\prod_i A_i^{s_i} \right)^2$$

where $\mathbf{s} = (s_i)$. Since \mathbf{s} is an alternating sum, all W -factors cancel on the left hand side of the equation. Therefore the left hand side is a product of the X_i , and the lemma follows by taking square roots on both sides. \square

We can now prove our formula for $\mathbb{I}_{\mathcal{D}}(l)$ in the case where l is an intersecting curve.

Proposition 6.5. Let l be a \mathcal{D} -lamination represented by a single intersecting curve of weight 1, and suppose that the orientation of each component of γ agrees with the orientation of S . Then

$$\mathbb{I}_{\mathcal{D}}(l) = \frac{\prod_{i \text{ even}} F_{c_i}(\hat{X}_1, \dots, \hat{X}_n)}{\prod_{i \text{ odd}} F_{c_i}(X_1, \dots, X_n)} B_1^{g_{l,1}} \dots B_n^{g_{l,n}} X_1^{h_{l,1}} \dots X_n^{h_{l,n}}$$

where the $g_{l,i}$ are integers and the $h_{l,i}$ are half integers.

Proof. By Lemma 6.3, we know that

$$A_{c_i} = F_{c_i}(X_1, \dots, X_n) A^{\mathbf{g}_{c_i}}$$

and

$$A_{c_i}^\circ = F_{c_i}(\hat{X}_1, \dots, \hat{X}_n) (A^\circ)^{\mathbf{g}_{c_i}}.$$

Inserting these expressions into the formula in the definition of $\mathbb{I}_{\mathcal{D}}$, we obtain

$$\mathbb{I}_{\mathcal{D}}(l) = \frac{\prod_{i \text{ even}} F_{c_i}(\hat{X}_1, \dots, \hat{X}_n)}{\prod_{i \text{ odd}} F_{c_i}(X_1, \dots, X_n)} \frac{(A^\circ)^{\mathbf{g}_{\text{even}}}}{A^{\mathbf{g}_{\text{odd}}}}$$

where we have defined $\mathbf{g}_{\text{even}} = \sum_{i \text{ even}} \mathbf{g}_{c_i}$ and $\mathbf{g}_{\text{odd}} = \sum_{i \text{ odd}} \mathbf{g}_{c_i}$. Substituting $A_i^\circ = B_i A_i$ into this expression, we obtain

$$\mathbb{I}_{\mathcal{D}}(l) = \frac{\prod_{i \text{ even}} F_{c_i}(\hat{X}_1, \dots, \hat{X}_n)}{\prod_{i \text{ odd}} F_{c_i}(X_1, \dots, X_n)} B^{\mathbf{g}_{\text{even}}} A^{\mathbf{g}_{\text{even}} - \mathbf{g}_{\text{odd}}}.$$

Finally, by Lemma 6.4, we can write the last factor as

$$A^{\mathbf{g}_{\text{even}} - \mathbf{g}_{\text{odd}}} = X^{\mathbf{h}}$$

for some half integral vector \mathbf{h} . This completes the proof. \square

Notice that the proof of Lemma 6.4 actually gives an explicit description of the product $X_1^{h_{l,1}} \dots X_n^{h_{l,n}}$ appearing in this theorem. In proving the lemma, we have essentially described a cycle η_l with $\mathbb{Z}/2\mathbb{Z}$ -coefficients on S such that this monomial equals $X_{i_1}^{\pm 1/2} \dots X_{i_s}^{\pm 1/2}$ where i_1, \dots, i_s are the edges of T that η_l intersects. This fact will be important below when we discuss rational functions obtained from laminations.

6.2.2 Other curves

We will now derive formulas for $\mathbb{I}_{\mathcal{D}}(c)$ in special cases where c is a closed curve on $S_{\mathcal{D}}$ that is not an intersecting curve. The following simple lemma will be used repeatedly for calculations.

Lemma 6.6. Let S be a decorated surface, T an ideal triangulation of S , and $m \in \mathcal{X}^+(S)$.

1. If c is a loop on S , then the monodromy around c is represented by a product of the matrices

$$\begin{pmatrix} X_i^{1/2} & 0 \\ X_i^{-1/2} & X_i^{-1/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_i^{1/2} & X_i^{1/2} \\ 0 & X_i^{-1/2} \end{pmatrix},$$

one for each edge i of T that the loop intersects.

2. If c is a loop surrounding a hole and the orientation of this hole agrees with the orientation of S , then the smallest eigenvalue of the monodromy around c is $X_{i_1}^{1/2} \dots X_{i_s}^{1/2}$ where i_1, \dots, i_s are the edges of the triangulation that spiral into this hole. If the orientation of the hole disagrees with the orientation of S , then $X_{i_1}^{-1/2} \dots X_{i_s}^{-1/2}$ is the smallest eigenvalue of the monodromy.

Proof. 1. This monodromy can be computed using the method described in Subsection 2.4.2. We first retract the curve c to the graph described there. If the resulting path turns to the left before crossing the edge i , then the monodromy includes a factor

$$\begin{pmatrix} 0 & X_i^{1/2} \\ -X_i^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} X_i^{1/2} & 0 \\ X_i^{-1/2} & X_i^{-1/2} \end{pmatrix} \in PSL(2, \mathbb{R}).$$

If it turns to the right before crossing i , then the monodromy includes a factor

$$\begin{pmatrix} 0 & X_i^{1/2} \\ -X_i^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} X_i^{1/2} & X_i^{1/2} \\ 0 & X_i^{-1/2} \end{pmatrix} \in PSL(2, \mathbb{R}).$$

The entire path can be decomposed into consecutive left and right turns.

2. By part 1, the monodromy around c is given by

$$\prod_{k=1}^s \begin{pmatrix} X_{i_k}^{1/2} & 0 \\ X_{i_k}^{-1/2} & X_{i_k}^{-1/2} \end{pmatrix} = \begin{pmatrix} X_{i_1}^{1/2} \dots X_{i_s}^{1/2} & 0 \\ C & X_{i_1}^{-1/2} \dots X_{i_s}^{-1/2} \end{pmatrix}$$

where C is a polynomial in the variables $X_{i_k}^{\pm 1/2}$. The eigenvalues of this matrix are the diagonal elements $X_{i_1}^{1/2} \dots X_{i_s}^{1/2}$ and $X_{i_1}^{-1/2} \dots X_{i_s}^{-1/2}$. If the orientation of the hole agrees with the orientation of S , then $X_{i_1} \dots X_{i_s} < 1$, and hence the first of these products is smaller. If the orientation of the hole disagrees with the orientation of S , then $X_{i_1} \dots X_{i_s} > 1$, and hence the second product is smaller. \square

Proposition 6.7. Let c be a \mathcal{D} -lamination consisting of a single loop of weight k that lies entirely in S° and is not retractible to the simple lamination. Then

$$\mathbb{I}_{\mathcal{D}}(c) = F_c(\widehat{X}_1, \dots, \widehat{X}_n) B_1^{g_{c,1}} \dots B_n^{g_{c,n}} X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$$

where F_c is a polynomial, the $g_{c,i}$ are integers, and the $h_{c,i}$ are half integers.

Proof. Let c be a \mathcal{D} -lamination satisfying the hypotheses of the proposition. In this case, $\mathbb{I}_{\mathcal{D}}(c)$ is defined as the absolute value of the trace of the k th power of the monodromy around c . By Lemma 6.6, this monodromy is a product of matrices

$$\begin{pmatrix} \widehat{X}_i^{1/2} & 0 \\ \widehat{X}_i^{-1/2} & \widehat{X}_i^{-1/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widehat{X}_i^{1/2} & \widehat{X}_i^{1/2} \\ 0 & \widehat{X}_i^{-1/2} \end{pmatrix},$$

one for each edge i that c intersects. These matrices factor as

$$\begin{pmatrix} \widehat{X}_i & 0 \\ 1 & 1 \end{pmatrix} \cdot \widehat{X}_i^{-1/2} \quad \text{and} \quad \begin{pmatrix} \widehat{X}_i & \widehat{X}_i \\ 0 & 1 \end{pmatrix} \cdot \widehat{X}_i^{-1/2},$$

so the monodromy factors as $M \cdot \widehat{X}_{i_1}^{-1/2} \dots \widehat{X}_{i_s}^{-1/2}$ where i_1, \dots, i_s are the edges that c intersects and M is a matrix with polynomial entries. Write $F_c(\widehat{X}_1, \dots, \widehat{X}_n)$ for the trace of the matrix M^k . Then we have

$$\mathbb{I}_{\mathcal{D}}(c) = F_c(\widehat{X}_1, \dots, \widehat{X}_n) \widehat{X}_{i_1}^{-k/2} \dots \widehat{X}_{i_s}^{-k/2}.$$

Consider the product $\prod_l \prod_j B_j^{\varepsilon_{lj}}$. Let i be any edge of the triangulation. Using the definition of the exchange matrix given in Section 2, it is easy to show that the total degree of B_i in this product is even. Hence

$$\begin{aligned} \widehat{X}_{i_1}^{-k/2} \dots \widehat{X}_{i_s}^{-k/2} &= \prod_l \left(X_{i_l} \prod_j B_j^{\varepsilon_{lj}} \right)^{-k/2} \\ &= \left(\prod_{j,l} B_j^{\varepsilon_{lj}} \right)^{-k/2} \left(\prod_l X_{i_l}^{-k/2} \right) \\ &= B_1^{g_{c,1}} \dots B_n^{g_{c,n}} X_1^{h_{c,1}} \dots X_n^{h_{c,n}} \end{aligned}$$

for some integers $g_{c,i}$ and half integers $h_{c,i}$. This completes the proof. \square

Proposition 6.8. Let c be a \mathcal{D} -lamination consisting of a single loop of weight k on $S_{\mathcal{D}}$. Assume that this loop is homotopic to a loop in the simple lamination and that c and m provide the same orientation for this loop. Then

$$\mathbb{I}_{\mathcal{D}}(c) = B_1^{g_{c,1}} \dots B_n^{g_{c,n}} X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$$

for integers $g_{c,i}$ and half integers $h_{c,i}$.

Proof. Let c be a \mathcal{D} -lamination satisfying the hypotheses of the proposition. In this case, $\mathbb{I}_{\mathcal{D}}(c)$ is defined as the absolute value of the k th power of the largest eigenvalue of the monodromy around c . Let i_1, \dots, i_s be the edges of the triangulation that spiral into this loop in the simple lamination. If the orientation provided by c agrees with the orientation of S° , then the orientation provided by m agrees with that of S° , and Lemma 6.6 implies

$$\mathbb{I}_{\mathcal{D}}(c) = \widehat{X}_{i_1}^{-1/2} \dots \widehat{X}_{i_s}^{-1/2}.$$

On the other hand, if the orientation provided by c agrees with the orientation of S , then the orientation provided by m disagrees with that of S° , and Lemma 6.6 implies

$$\mathbb{I}_{\mathcal{D}}(c) = \widehat{X}_{i_1}^{1/2} \dots \widehat{X}_{i_s}^{1/2}.$$

The same argument that we used in the proof of Proposition 6.7 can be used to show that the k th powers of these expressions have the form $B_1^{g_{c,1}} \dots B_n^{g_{c,n}} X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$ for some integers $g_{c,i}$ and half integers $h_{c,i}$. \square

Proposition 6.9. Let c be a \mathcal{D} -lamination consisting of a single loop of weight k that lies entirely in S and is not retractible to the simple lamination. Then

$$\mathbb{I}_{\mathcal{D}}(c) = \frac{1}{F_c(X_1, \dots, X_n)} X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$$

where F_c is a polynomial and the $h_{c,i}$ are half integers.

Proof. Let c be a \mathcal{D} -lamination satisfying the hypotheses of the proposition. In this case, $\mathbb{I}_{\mathcal{D}}(c)$ is defined as the reciprocal of the absolute value of the trace of the k th power of the monodromy around c . By Lemma 6.6, this monodromy is an element of $PSL(2, \mathbb{R})$ represented by a product of the matrices

$$\begin{pmatrix} X_i^{1/2} & 0 \\ X_i^{-1/2} & X_i^{-1/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_i^{1/2} & X_i^{1/2} \\ 0 & X_i^{-1/2} \end{pmatrix},$$

one for each edge i that c intersects. These matrices factor as

$$\begin{pmatrix} X_i & 0 \\ 1 & 1 \end{pmatrix} \cdot X_i^{-1/2} \quad \text{and} \quad \begin{pmatrix} X_i & X_i \\ 0 & 1 \end{pmatrix} \cdot X_i^{-1/2},$$

so the monodromy factors as $M \cdot X_{i_1}^{-1/2} \dots X_{i_s}^{-1/2}$ where i_1, \dots, i_s are the edges that c intersects and M is a matrix with polynomial entries. Write $F_c(X_1, \dots, X_n)$ for the trace of the matrix M^k . Then we have

$$\begin{aligned} \mathbb{I}_{\mathcal{D}}(c) &= \frac{1}{F_c(X_1, \dots, X_n)} X_{i_1}^{k/2} \dots X_{i_s}^{k/2} \\ &= \frac{1}{F_c(X_1, \dots, X_n)} X_1^{h_{c,1}} \dots X_n^{h_{c,n}} \end{aligned}$$

for half integers $h_{c,i}$. \square

Proposition 6.10. Let c be a \mathcal{D} -lamination consisting of a single loop of weight k on $S_{\mathcal{D}}$. Assume that this loop is homotopic to a loop in the simple lamination and that c and m provide different orientations for this loop. Then

$$\mathbb{I}_{\mathcal{D}}(c) = X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$$

for half integers $h_{c,i}$.

Proof. Let c be a \mathcal{D} -lamination satisfying the hypotheses of the proposition. In this case, $\mathbb{I}_{\mathcal{D}}(c)$ is defined as the inverse absolute value of the k th power of an eigenvalue of the monodromy around c . Let i_1, \dots, i_s be the edges of the triangulation that spiral into this loop in the simple lamination. If the orientation provided by c agrees with that of S , then the orientation provided by m disagrees with S , and Lemma 6.6 implies

$$\mathbb{I}_{\mathcal{D}}(c) = X_{i_1}^{-1/2} \dots X_{i_s}^{-1/2}.$$

On the other hand, if the orientation provided by c agrees with the orientation of S° , then the orientation provided by m agrees with S , and Lemma 6.6 implies

$$\mathbb{I}_{\mathcal{D}}(c) = X_{i_1}^{1/2} \dots X_{i_s}^{1/2}.$$

Thus $\mathbb{I}_{\mathcal{D}}(c)$ has the desired form. \square

Propositions 6.7, 6.8, 6.9, and 6.10 provide expressions for $\mathbb{I}_{\mathcal{D}}(c)$ in the cases where the curve c is not an intersecting curve. In each case, the expression includes a product of the form $X_1^{h_{c,1}} \dots X_n^{h_{c,n}}$ where each $h_{c,i}$ is a half integer. In fact, the proofs of these propositions provide a more explicit description of this product: It is obtained by multiplying one factor $X_i^{\pm k/2}$ each time the curve c crosses an edge i . This fact will be important below when we discuss rational functions obtained from laminations.

6.2.3 The general case

We will now combine the above results to get a general expression for the function $\mathbb{I}_{\mathcal{D}}(l)$. To state this result, we will need the following notation. Any \mathcal{D} -lamination $l \in \mathcal{D}_L(S, \mathbb{Z})$ can be represented by a collection of curves of weight 1. If this collection contains homotopic curves of weights a and b which are not intersecting curves, let us replace these by a single curve of weight $a + b$. In this way, we obtain a new collection of curves representing l . If we now cut the surface along the image of ∂S , we obtain a collection \mathcal{C}° of curves on S° and a collection \mathcal{C} of curves on S .

Theorem 6.11. Let $l \in \mathcal{D}_L(S, \mathbb{Z})$ and suppose that the orientation of the curve γ_i agrees with S whenever γ_i meets a curve of l . Then

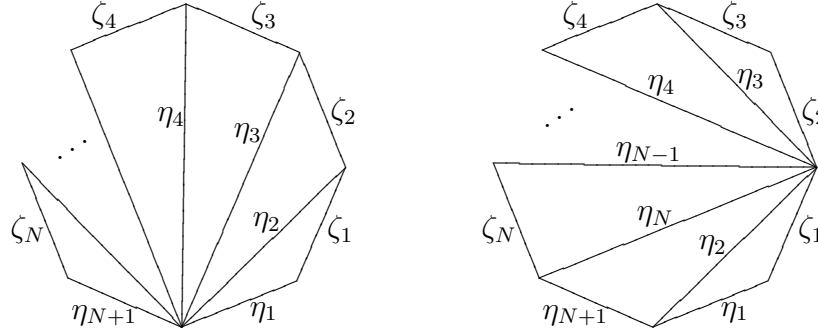
$$\mathbb{I}_{\mathcal{D}}(l) = \frac{\prod_{c \in \mathcal{C}^\circ} F_c(\widehat{X}_1, \dots, \widehat{X}_n)}{\prod_{c \in \mathcal{C}} F_c(X_1, \dots, X_n)} B_1^{g_{l,1}} \dots B_n^{g_{l,n}} X_1^{h_{l,1}} \dots X_n^{h_{l,n}}$$

where the F_c are polynomials, the $g_{l,i}$ are integers and the $h_{l,i}$ are half integers.

Proof. If c is a closed curve which is homotopic to a loop in the simple lamination, we set $F_c = 1$. Then the above formula for $\mathbb{I}_{\mathcal{D}}(l)$ is an immediate consequence of Propositions 6.7, 6.8, 6.9, 6.10, and 6.5 and the multiplicativity of $\mathbb{I}_{\mathcal{D}}$. \square

Proposition 6.12. The function κ_i in Definition 6.2 is given by a subtraction-free rational expression in the B_j and X_j ($j \in J$).

Proof. If the expressions α and α° each have only one term, then we can obviously express κ_i as a Laurent polynomial in the B_j . We will therefore assume that these expressions have more than one term. Consider the ideal polygon P , illustrated below on the left, which we used to define κ_i . This polygon has a triangulation $T(P)$ coming from the triangulation of the surface. Consider the triangulation $T'(P)$ illustrated on the right. We can transform this new triangulation back to $T(P)$ by performing flips at the edges labeled $\eta_N, \eta_{N-1}, \dots, \eta_3$ in the diagram on the right in this order.



Let us write A'_i for the invariant associated to an edge i of $T'(P)$. Note that by the cluster mutation formula, we have

$$\begin{aligned} A'_{\eta_N} A_{\eta_3} A_{\eta_4} \dots A_{\eta_N} &= (A'_{\eta_N} A_{\eta_N}) (A_{\eta_3} A_{\eta_4} \dots A_{\eta_{N-1}}) \\ &= (A'_{\eta_{N-1}} A_{\eta_{N+1}} + A_{\zeta_N} A_{\eta_2}) (A_{\eta_3} A_{\eta_4} \dots A_{\eta_{N-1}}) \\ &= (A'_{\eta_{N-1}} A_{\eta_{N-1}}) (A_{\eta_3} A_{\eta_4} \dots A_{\eta_{N-2}} A_{\eta_{N+1}}) \\ &\quad + A_{\zeta_N} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-1}}. \end{aligned}$$

Applying the same formula again, we see that

$$\begin{aligned} A'_{\eta_N} A_{\eta_3} A_{\eta_4} \dots A_{\eta_N} &= (A'_{\eta_{N-2}} A_{\eta_N} + A_{\zeta_{N-1}} A_{\eta_2}) (A_{\eta_3} A_{\eta_4} \dots A_{\eta_{N-2}} A_{\eta_{N+1}}) \\ &\quad + A_{\zeta_N} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-1}} \\ &= (A'_{\eta_{N-2}} A_{\eta_{N-2}}) (A_{\eta_3} \dots A_{\eta_{N-3}} A_{\eta_N} A_{\eta_{N+1}}) \\ &\quad + A_{\zeta_{N-1}} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-2}} A_{\eta_{N+1}} \\ &\quad + A_{\zeta_N} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-1}}. \end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned} A'_{\eta_N} A_{\eta_3} A_{\eta_4} \dots A_{\eta_N} &= A_{\zeta_2} A_{\eta_4} A_{\eta_5} \dots A_{\eta_{N+1}} + \dots \\ &\quad + A_{\zeta_{N-1}} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-2}} A_{\eta_{N+1}} \\ &\quad + A_{\zeta_N} A_{\eta_2} A_{\eta_3} \dots A_{\eta_{N-1}}. \end{aligned}$$

Consider the cross ratio

$$X'_{\eta_2} = \frac{A'_{\eta_N} A_{\eta_1}}{A_{\zeta_1} A_{\eta_{N+1}}}.$$

By the previous calculation, we have

$$\begin{aligned}
(1 + X'_{\eta_2})A_{\zeta_1}A_{\eta_3} \dots A_{\eta_{N+1}} &= A_{\zeta_1}A_{\eta_3} \dots A_{\eta_{N+1}} + A'_{\eta_N}A_{\eta_1}A_{\eta_3}A_{\eta_4} \dots A_{\eta_N} \\
&= A_{\zeta_1}A_{\eta_3} \dots A_{\eta_{N+1}} + A_{\zeta_2}A_{\eta_1}A_{\eta_4}A_{\eta_5} \dots A_{\eta_{N+1}} + \dots \\
&\quad + A_{\zeta_{N-1}}A_{\eta_1}A_{\eta_2} \dots A_{\eta_{N-2}}A_{\eta_{N+1}} \\
&\quad + A_{\zeta_N}A_{\eta_1}A_{\eta_2} \dots A_{\eta_{N-1}}
\end{aligned}$$

and hence

$$(1 + X'_{\eta_2}) \frac{A_{\zeta_1}A_{\eta_3} \dots A_{\eta_{N+1}}}{A_{\eta_1} \dots A_{\eta_{N+1}}} = \sum_{i=1}^N \frac{A_{\zeta_i}}{A_{\eta_i}A_{\eta_{i+1}}}.$$

Now the triangulation $T'(P)$ determines a similar triangulated polygon in the universal cover of S° . Write $(A'_i)^\circ$ for the invariant associated to an edge i of this triangulation. Then a similar argument shows that the cross ratio $\widehat{X}'_{\eta_2} = \frac{(A'_{\eta_N})^\circ A_{\eta_1}^\circ}{A_{\zeta_1}^\circ A_{\eta_{N+1}}^\circ}$ satisfies

$$(1 + \widehat{X}'_{\eta_2}) \frac{A_{\zeta_1}^\circ A_{\eta_3}^\circ \dots A_{\eta_{N+1}}^\circ}{A_{\eta_1}^\circ \dots A_{\eta_{N+1}}^\circ} = \sum_{i=1}^N \frac{A_{\zeta_i}^\circ}{A_{\eta_i}^\circ A_{\eta_{i+1}}^\circ}.$$

Dividing, we obtain an expression for κ_i :

$$\kappa_i = \frac{1 + \widehat{X}'_{\eta_2}}{1 + X'_{\eta_2}} B_{\zeta_1} B_{\eta_1}^{-1} B_{\eta_2}^{-1}.$$

This implies the proposition since we can express \widehat{X}'_{η_2} and X'_{η_2} as rational functions of the B - and X -coordinates. \square

It follows from the above lemma that κ_i is independent of the horocycles used in the construction as well as the chosen polygon.

6.2.4 Expansion of cluster variables

Theorem 6.11 is an extension of Proposition 5.8, which gives a formula expressing the B -variables of the cluster \mathcal{D} -algebra in terms of the B - and X -variables of an initial seed.

To see this, suppose T is an ideal triangulation of S with no self-folded triangles. This triangulation determines an initial seed for the cluster algebra \mathcal{D} . Let c be an arc of some other triangulation with no self-folded triangles. By gluing this arc to its image under the map $S \rightarrow S^\circ$, we get an intersecting curve l on $S_{\mathcal{D}}$. The canonical function $\mathbb{I}_{\mathcal{D}}(l)$ associated to this intersecting curve is simply the B -coordinate B_c corresponding to the arc c . Suppose there is a sequence of regular flips taking the triangulation T to the triangulation containing c . By Proposition 5.8, we can write

$$B_c = \frac{F_c(\widehat{X}_1, \dots, \widehat{X}_n)}{F_c(X_1, \dots, X_n)} B_1^{g_1} \dots B_n^{g_n}$$

where the B_j and X_j are the coordinates associated to the internal edges of T . The F -polynomials in this formula can be computed using the matrix formulas of Musiker and Williams [MW]. Since T has no self-folded triangles, one gets the same polynomials by lifting the arc c and triangulation T to the universal cover and applying the results of Musiker and Williams there. This implies that the F -polynomials appearing in this formula are the same ones we get from Theorem 6.11.

6.2.5 Generalized F -polynomials

Notice that if c is any arc connecting loops in the simple lamination on $S_{\mathcal{D}}$, then the polynomial F_c appearing in Theorem 6.11 is an F -polynomial in the sense of Fomin and Zelevinsky. On the other hand, if c is a closed loop, then F_c does not arise in this way. This suggests that the polynomials appearing in this theorem should be seen as generalizing the F -polynomials. Such generalized F -polynomials have appeared previously in the work of Musiker, Schiffler, and Williams [MW, MSW2] where they are defined in terms of perfect matchings of a graph.

Indeed, suppose T is an ideal triangulation of S with no self-folded triangles. It follows from Lemma 2.13 of [FST] that such a triangulation always exists for the surfaces that Musiker and Williams consider in [MW]. Let c be a loop on $S_{\mathcal{D}}$ that lies entirely in S and is not homotopic to a loop in the simple lamination. The monodromy around such a loop was computed in the proof of Proposition 6.9 as a product of matrices. By Proposition 4.12 of [MW], the absolute value of the trace of this monodromy is given by their expression $\bar{\chi}_{c,T}$ with the substitutions $x_i = 1$ and $y_i = X_i$. Using the notation from [MW], we have

$$\begin{aligned} F_c(X_1, \dots, X_n) &= \mathbb{I}_{\mathcal{D}}(c)^{-1} X_{i_1}^{1/2} \dots X_{i_s}^{1/2} \\ &= \bar{\chi}_{c,T} X_{i_1}^{1/2} \dots X_{i_s}^{1/2} \\ &= \widehat{\chi}_{c,T} \end{aligned}$$

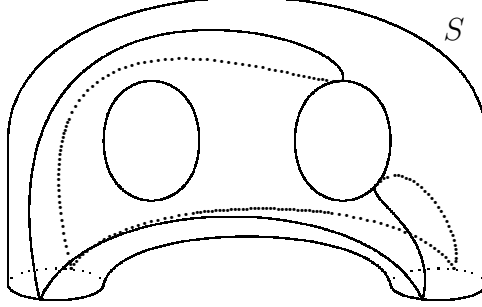
with $x_i = 1$ and $y_i = X_i$. For a triangulation without self-folded triangles, Musiker and Williams show that this last expression equals the generalized F -polynomial that they associate to the curve c .

If the curve c lies on S° instead of S , then the same argument with the variables X_i replaced by \widehat{X}_i shows that our expressions $F_c(\widehat{X}_1, \dots, \widehat{X}_n)$ agree with the generalized F -polynomials introduced by Musiker, Schiffler, and Williams.

6.3 Rational functions from laminations

Theorem 6.11 allows us to completely characterize the function $\mathbb{I}_{\mathcal{D}}(l)$ associated to a \mathcal{D} -lamination $l \in \mathcal{D}_L(S, \mathbb{Z})$. However, the exponents h_i are only half integral, so an element of $\mathcal{D}_L(S, \mathbb{Z})$ does not give rise to a rational function in the B_i and X_i in general. In analogy with the classical case of \mathcal{A} - and \mathcal{X} -laminations, we expect that $\mathbb{I}_{\mathcal{D}}(l)$ is a rational function of the variables B_i and X_i if and only if l has integral coordinates. This expectation turns out to be correct, and we will prove it in several steps. In addition, we will give a homological condition on a \mathcal{D} -lamination l which is equivalent to the integrality of the coordinates of l .

To state the relevant homological condition, we first use the natural map $S^\circ \rightarrow S$ to draw all of the curves in \mathcal{C} or \mathcal{C}° on the surface S . For example, the intersecting curve illustrated in the introduction gives the following picture.



In this way, we get a collection of curves on S , and any arc in this collection that connects two holes can be viewed as a singular 1-simplex with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. Similarly, if c is a closed loop in this collection, then we can view this loop as a 1-cycle with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. This cycle consists of a single 0-simplex in the interior of each triangle that c intersects in a fixed ideal triangulation T together with a 1-simplex connecting each pair of consecutive 0-simplices. Let σ_l be sum of all the 1-simplices obtained in this way from arcs and closed loops. Then σ_l is a cycle representing an element $[\sigma_l]$ of the singular homology $H_1(S, \mathbb{Z}/2\mathbb{Z})$ of S with coefficients in $\mathbb{Z}/2\mathbb{Z}$. With this notation, one has the following result.

Theorem 6.13. Let $l \in \mathcal{D}_L(S, \mathbb{Z})$ be a \mathcal{D} -lamination. The following are equivalent:

1. $b_j, x_j \in \mathbb{Z}$ for all $j \in J$.
2. $[\sigma_l] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$.
3. $\mathbb{I}_{\mathcal{D}}(l)$ is a rational function in the variables B_j and X_j .

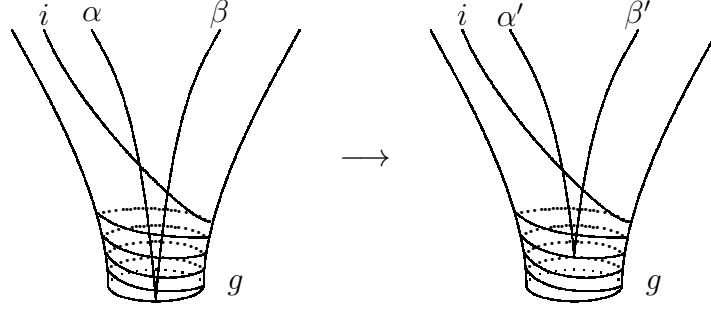
Proof. To simplify notation, we will denote σ_l simply by σ . We assume that the ideal triangulation T has been drawn on S in such a way that the edges spiral into the holes of S in the direction specified by l .

Step 1. $1 \implies 2$

Suppose $l \in \mathcal{D}_L(S, \mathbb{Z})$ is a lamination with integral coordinates. The 1-simplices that make up σ may intersect the triangulation in infinitely many points. We will begin by constructing a homologous cycle σ' that intersects each edge of the triangulation in a finite, and in fact an *even*, number of points. We will then show that $[\sigma'] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$.

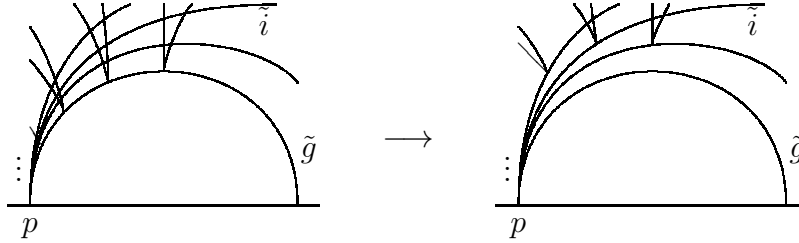
To define this cycle σ' , consider any hole in S that meets one of the simplices of σ . Modify the cycle σ near this hole as illustrated below so that the simplices meet at an edge of the triangulation spiraling into the hole rather than at a point of ∂S . Doing this near every hole

gives the desired cycle σ' .



It is easy to see that σ' is homologous to σ . We claim that it intersects each edge of the triangulation of S in an even number of points. (Here the number of intersections is counted with multiplicity so that if two 1-simplices intersect an edge i at the same point, then we say there are two intersections.)

Indeed, consider the preimages of σ and σ' in the universal cover \tilde{S} of S .



Let g be the boundary of a hole in S , and let \tilde{g} be an arc in the universal cover that projects to g . Then the preimage of σ near g consists of infinitely many curves meeting \tilde{g} as illustrated above on the left. These curves are alternately preimages of α or β . By definition, the coordinate b_i is obtained by choosing, near each endpoint p of \tilde{i} , a distinguished pair of preimages $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ of α and β , respectively. We then count the number n_i of preimages of curves from S that intersect \tilde{i} between the distinguished curves and the number n_i° of preimages of curves from S° that intersect \tilde{i} between the distinguished curves. Then b_i is given by

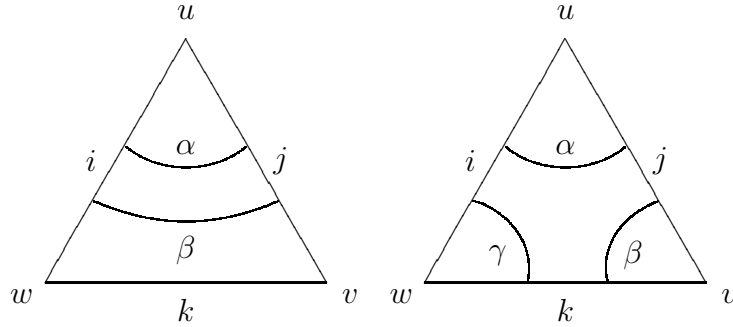
$$b_i = \frac{n_i^\circ}{2} - \frac{n_i}{2}.$$

Since we are assuming $b_i \in \mathbb{Z}$ for all i , we know that $n_i^\circ - n_i \in 2\mathbb{Z}$, or equivalently that $n_i^\circ + n_i \in 2\mathbb{Z}$. When we modify σ to get the cycle σ' , the preimage of σ in \tilde{S} is modified as in the illustration above. The distinguished curves $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ that we chose above determine preimages $\tilde{\alpha}'_p$ and $\tilde{\beta}'_p$ of α' and β' , respectively. As before, we can count the number of intersections between \tilde{i} and the preimages of 1-simplices of σ' . The total number of such intersections differs from $n_i^\circ + n_i$ by an even number. Hence the total number of these intersections is even. But this number equals the number of intersections of σ' with the edge i . This proves that σ' intersects each edge of the triangulation in an even number of points.

It remains to show that $[\sigma'] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. If σ_c is the summand of σ' obtained from a closed loop c on S or S° , we can modify this summand to get a homologous cycle whose

vertices all lie on edges of T . In general, the simplices of the resulting cycle may intersect, and we will deform these arcs so that each of the intersections occurs on an edge of the ideal triangulation. We will consider the refinement of the resulting cycle whose 1-simplices are the intersections of the 1-simplices with the triangles of the ideal triangulation. Abusing notation, we will denote this refinement also by σ' . We want to show that this σ' represents the zero class in homology.

Consider a triangle on the surface with sides i , j , and k . The simplices of σ' contained in this triangle are arcs connecting adjacent sides of the triangle so that the total number of arcs that terminate on a given side is always even. Let α be an arc joining two sides, say i and j . We may assume that there are no additional arcs between α and the vertex u that it surrounds.



There is an even number of arcs that meet the edge j . It follows that α cannot be the only one and hence there is another arc β that terminates on this edge. By our choice of α , this arc must not lie between α and u , so β looks like one of the arcs illustrated above depending on whether it ends on i or k .

First consider the case where β ends on i . In this case, there is a cycle formed by the arcs α and β together with the portions of i and j between the endpoints of these arcs. This cycle is obviously zero in $H_1(S, \mathbb{Z}/2\mathbb{Z})$.

Now consider the case where β joins the edges j and k . We can assume that there are no arcs between β and the vertex v that it surrounds. Since the total number of arcs that meet the edge k is even, there is another arc γ that meets k . By our choice of β , this arc must not lie between β and v . Moreover, we can assume that it does not join k and j because this case was already treated above. It follows that γ joins k and i as illustrated above. There is a cycle formed by the arcs α , β , and γ together with the portions of i , j , and k between their endpoints. This cycle is again zero in $H_1(S, \mathbb{Z}/2\mathbb{Z})$.

Thus we define for any triangle a cycle with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. Consider the sum of this cycle with σ' in the space of all such cycles. It is another cycle representing the same homology class as σ' . It consists of arcs on the edges of the triangulation together with arcs connecting different edges, and the total number of arcs of the latter type that meet at a given edge is always even. Thus we can iterate the above construction and continue adding cycles to get new cycles homologous to σ' . At each step, the number of arcs connecting different edges decreases, so eventually we are left with zero. This proves that σ' is homologous to zero.

Step 2. $2 \implies 1$

Suppose our cycle satisfies $[\sigma] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. Since $l \in \mathcal{D}_L(S, \mathbb{Z})$, we can realize l as a collection of curves on $S_{\mathcal{D}}$ with integral weights. Then the portion of l that lies on $S \subseteq S_{\mathcal{D}}$ is a collection of curves with integral weights. Since $\mathcal{X}_L(S, \mathbb{Z}) = \mathcal{X}(\mathbb{Z}^t)$, it follows that l has integral x -coordinates. Thus we only need to show that l has integral b -coordinates.

To prove this fact, we first modify σ as in Step 1 to get a new cycle σ' . In general, the interiors of two 1-simplices of σ' may intersect. If such an intersection point lies on an edge i of T , let us deform this edge slightly to avoid the intersection without changing the number of intersections between i and the interior points of 1-simplices of σ' . We then take a refinement of σ' in which every point of self intersection is a 0-simplex. Abusing notation, we will denote this refinement also by σ' .

Since σ' is homologous to σ , our assumption implies $[\sigma'] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. We can realize S as a two-dimensional simplicial complex Δ such that every simplex of σ' is included in Δ . This simplicial complex defines a simplicial homology group $H_1^\Delta(S, \mathbb{Z}/2\mathbb{Z})$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. There is an isomorphism

$$H_1^\Delta(S, \mathbb{Z}/2\mathbb{Z}) \cong H_1(S, \mathbb{Z}/2\mathbb{Z})$$

between simplicial and singular homology, and so σ' also represents the zero class in *simplicial* homology. This means there is a chain η of simplices in Δ with $\sigma' = \partial\eta$. Since σ' does not meet a marked point or hole on S , we know that η does not meet a marked point or hole S . It follows that the endpoints of an edge i of T cannot lie on η . If i crosses σ' into the interior of a 2-simplex of η , then it must leave η by crossing σ' again.

It follows that i intersects σ' in an even number of points. As we showed in Step 1, this number of intersections differs by an even number from $n_i^\circ + n_i$, so we have $n_i^\circ + n_i \in 2\mathbb{Z}$ and hence $n_i^\circ - n_i \in 2\mathbb{Z}$. Hence $b_i = \frac{1}{2}(n_i^\circ - n_i) \in \mathbb{Z}$ as desired.

Step 3. $2 \implies 3$

Suppose the cycle σ satisfies $[\sigma] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. As in Steps 1 and 2, we will begin by replacing σ by a homologous cycle.

Let c be a closed curve in \mathcal{C} or \mathcal{C}° . Then Propositions 6.7, 6.8, 6.9, and 6.10 give an expression for $\mathbb{I}_{\mathcal{D}}(c)$ in terms of the B - and X -coordinates. This expression includes a factor which is a monomial in the X -coordinates with half integral exponents. By the remarks following Proposition 6.10, we know that this monomial has the form $X_{i_1}^{\pm k/2} \dots X_{i_s}^{\pm k/2}$ where i_1, \dots, i_s are the edges of T that σ_c intersects. Similarly, if c is an intersecting curve on $S_{\mathcal{D}}$, then Proposition 6.5 gives an expression for $\mathbb{I}_{\mathcal{D}}(c)$ in terms of the B - and X -coordinates. This expression includes a factor which is a monomial in the X -coordinates with half integral exponents. In proving Proposition 6.5, we essentially constructed a cycle σ'_c with $\mathbb{Z}/2\mathbb{Z}$ -coefficients such that this monomial has the form $X_{i_1}^{\pm 1/2} \dots X_{i_s}^{\pm 1/2}$ where i_1, \dots, i_s are the edges of T that σ'_c intersects.

It is easy to see that σ'_c is homologous to σ_c for any intersecting curve c . Thus for an arbitrary \mathcal{D} -lamination l , we can replace σ by a homologous cycle σ' in which every summand of σ of the form σ_c for c an intersecting curve is replaced by σ'_c . Our assumption

that $[\sigma] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ implies that $[\sigma'] = 0$ as well. Arguing as in Step 2, we find that each edge of T intersects σ' in an even number of points.

Now Theorem 6.11 gives an expression for $\mathbb{I}_{\mathcal{D}}(l)$ in terms of B - and X -coordinates for any \mathcal{D} -lamination l . This expression includes a factor which is a monomial in the X -coordinates with half integral exponents $h_{l,i}$, and we want to show that each $h_{l,i}$ is in fact an integer. Since $\mathbb{I}_{\mathcal{D}}(l)$ is the product of the functions $\mathbb{I}_{\mathcal{D}}(c)$ where c is an intersecting curve in l or a loop in \mathcal{C} or \mathcal{C}° , this follows immediately from the fact that any edge i of the triangulation T intersects σ' in an even number of points.

Step 4. $3 \implies 2$

Suppose $\mathbb{I}_{\mathcal{D}}(l)$ is a rational function in the variables B_j and X_j . Then each of the exponents $h_{l,i}$ in Theorem 6.11 is an integer. This means that each edge of the ideal triangulation T intersects the cycle σ' constructed in Step 3 in an even number of points. Arguing as in Step 1, we can recursively add cycles to σ' to prove that $[\sigma'] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. Since σ' is homologous to σ , this implies $[\sigma] = 0 \in H_1(S, \mathbb{Z}/2\mathbb{Z})$. \square

6.4 Further properties

Note that there is a natural map $\varphi : \mathcal{A}_0 \times \mathcal{A}_0 \rightarrow \mathcal{D}$ taking the product of two copies of the cluster \mathcal{A} -variety into the symplectic double. It is given in coordinates by the formulas

$$\begin{aligned}\varphi^*(B_i) &= \frac{A_i^\circ}{A_i}, \\ \varphi^*(X_i) &= \prod_j A_j^{\varepsilon_{ij}}.\end{aligned}$$

where A_i° are the coordinates on the second copy of \mathcal{A}_0 . This map induces maps $\varphi : \mathcal{A}_0^+ \times \mathcal{A}_0^+ \rightarrow \mathcal{D}^+$ and $\varphi : \mathcal{A}_0(\mathbb{Z}^t) \times \mathcal{A}_0(\mathbb{Z}^t) \rightarrow \mathcal{D}(\mathbb{Z}^t)$, which we denote by the same symbol φ . Concretely, these maps are defined by gluing together hyperbolic surfaces or laminations, respectively.

There is also a natural projection $\pi : \mathcal{D} \rightarrow \mathcal{X} \times \mathcal{X}$ given in coordinates by the formulas

$$\begin{aligned}\varphi^*(X_i) &= X_i, \\ \varphi^*(\widehat{X}_i) &= X_i \prod_j B_j^{\varepsilon_{ij}}.\end{aligned}$$

where \widehat{X}_i are the coordinates on the second copy of \mathcal{X} . This projection induces maps $\pi : \mathcal{D}^+ \rightarrow \mathcal{X}^+ \times \mathcal{X}^+$ and $\pi : \mathcal{D}(\mathbb{Z}^t) \rightarrow \mathcal{X}(\mathbb{Z}^t) \times \mathcal{X}(\mathbb{Z}^t)$, which we denote by the same symbol π . Concretely, these maps are defined by cutting the surface $S_{\mathcal{D}}$ to recover the surfaces S and S° .

We will write $\text{Fun}(\mathcal{D}^+)$ for the set of all functions $\mathcal{D}^+ \rightarrow \mathbb{R}_{>0}$ that are rational in the B - and X -coordinates. We will likewise denote by $\text{Fun}(\mathcal{A}_0^+ \times \mathcal{A}_0^+)$ the set of all functions $\mathcal{A}_0^+ \times \mathcal{A}_0^+ \rightarrow \mathbb{R}_{>0}$ that are rational in the coordinates on the two copies of \mathcal{A}_0^+ , and we will

denote by $\text{Fun}(\mathcal{X}^+ \times \mathcal{X}^+)$ the set of all functions $\mathcal{X}^+ \times \mathcal{X}^+ \rightarrow \mathbb{R}_{>0}$ that are rational in the coordinates on the two copies of \mathcal{X}^+ . We have seen that the multiplicative canonical pairing provides a map

$$\mathbb{I}_{\mathcal{D}} : \mathcal{D}(\mathbb{Z}^t) \rightarrow \text{Fun}(\mathcal{D}^+).$$

There is a map $\mathbb{I}_{\mathcal{A}_0 \times \mathcal{A}_0} : \mathcal{A}_0(\mathbb{Z}^t) \times \mathcal{A}_0(\mathbb{Z}^t) \rightarrow \text{Fun}(\mathcal{X}^+ \times \mathcal{X}^+)$ which takes a pair (l, l°) of laminations to the function $\mathbb{I}(l^\circ)/\mathbb{I}(l)$. There is a similar map $\mathbb{I}_{\mathcal{X} \times \mathcal{X}} : \mathcal{X}(\mathbb{Z}^t) \times \mathcal{X}(\mathbb{Z}^t) \rightarrow \text{Fun}(\mathcal{A}_0^+ \times \mathcal{A}_0^+)$ which takes a pair (l, l°) to the function $\mathbb{I}(l^\circ)/\mathbb{I}(l)$.

The following proposition shows that the map $\mathbb{I}_{\mathcal{D}}$ is naturally compatible with the classical canonical pairings. It follows immediately from the definitions.

Proposition 6.14. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_0(\mathbb{Z}^t) \times \mathcal{A}_0(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{A}_0 \times \mathcal{A}_0}} & \text{Fun}(\mathcal{X}^+ \times \mathcal{X}^+) \\ \varphi \downarrow & & \downarrow \pi^* \\ \mathcal{D}(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{D}}} & \text{Fun}(\mathcal{D}^+) \\ \pi \downarrow & & \downarrow \varphi^* \\ \mathcal{X}(\mathbb{Z}^t) \times \mathcal{X}(\mathbb{Z}^t) & \xrightarrow{\mathbb{I}_{\mathcal{X} \times \mathcal{X}}} & \text{Fun}(\mathcal{A}_0^+ \times \mathcal{A}_0^+). \end{array}$$

In [FG1], Fock and Goncharov proved a number of special properties of the classical canonical pairings. In particular, they showed that the canonical functions provided by these pairing are expressible in any coordinate system as Laurent polynomials with positive integral coefficients. By Theorem 6.11, Proposition 6.12, and Theorem 6.13, we see that each function $\mathbb{I}_{\mathcal{D}}(l)$ for $l \in \mathcal{D}(\mathbb{Z}^t)$ can be expressed as a subtraction-free rational function in the B - and X -coordinates.

One of the main results of [FG1] states that the Laurent polynomials $\mathbb{I}(l)$ for $l \in \mathcal{A}(\mathbb{Z}^t)$ an \mathcal{A} -lamination on a punctured surface provide a canonical vector space basis for the algebra of regular functions on the cluster \mathcal{X} -variety. This basis has the special property that the structure constants for multiplication of basis elements are positive integers. Since the functions $\mathbb{I}_{\mathcal{D}}(l)$ defined in the present paper for $l \in \mathcal{D}(\mathbb{Z}^t)$ are *rational* rather than regular functions, we do not have a similar canonical basis parametrized by \mathcal{D} -laminations. Understanding the precise relationship between the construction described here and canonical bases is an interesting problem for future research.

6.5 The intersection pairing

In addition to the multiplicative canonical pairing $\mathbb{I}_{\mathcal{D}} : \mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}^+(S) \rightarrow \mathbb{R}_{>0}$, we have a canonical map $\mathcal{I}_{\mathcal{D}} : \mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}_L(S, \mathbb{Z}) \rightarrow \mathbb{Q}$ defined as follows.

Suppose we are given an intersecting curve l on $S_{\mathcal{D}}$. Deform this curve so that it intersects the image of ∂S in the minimal number of points. Then, starting from a component of this image which we may call γ_1 , there is a segment c_1 of the curve l which lies entirely in S and connects the component γ_1 to another component which we may call γ_2 . Starting from

this component, there is a segment c_2 of l which lies entirely in S° and connects γ_2 to a component γ_3 . Continue labeling in this way until the curve closes.

Now suppose we are given a point $m \in \mathcal{D}_L(S, \mathbb{Z})$. Draw this lamination on the same copy of $S_{\mathcal{D}}$ as l and deform its curves so that they intersect the image of ∂S in the minimal number of points. Cut the surface $S_{\mathcal{D}}$ along ∂S . Then each c_i is a curve that connects two boundary components of S or S° . We can wind the ends of c_i around the holes infinitely many times in the direction prescribed by the orientations from m .

Consider a curve c_i for i odd. Lifting this curve to the universal cover of S , we obtain a geodesic \tilde{c}_i connecting two points on the boundary of \mathbb{H} . Choose a distinguished curve intersecting \tilde{c}_i near each of these boundary points, and define a_{c_i} as half the number of intersections of the lifted curves between the distinguished curves. Next consider c_i for i even. Lifting to the universal cover of S° , we again get a geodesic connecting two points on the boundary of \mathbb{H} , and the distinguished curves already chosen determine a pair of distinguished curves near these points. We define $a_{c_i}^\circ$ as half the number of intersections of the lifted curves between the distinguished curves.

Definition 6.15. Fix a point $m \in \mathcal{D}_L(S, \mathbb{Z})$.

1. Let l be an intersecting curve of weight k on $S_{\mathcal{D}}$, and assume that the orientation of each component of γ agrees with the orientation of S . Then

$$\mathcal{I}_{\mathcal{D}}(l, m) = k \left(\sum_{i \text{ even}} a_{c_i}^\circ - \sum_{i \text{ odd}} a_{c_i} \right).$$

2. Let l be a curve of weight k on $S_{\mathcal{D}}$ which is not an intersecting curve and is not homotopic to a loop in the simple lamination. If l lies in S° , then $\mathcal{I}_{\mathcal{D}}(l, m)$ is defined as $k/2$ times the minimal number of intersections between l and m . If l lies in S then $\mathcal{I}_{\mathcal{D}}(l, m)$ is minus this quantity.
3. Let l be a curve of weight k on $S_{\mathcal{D}}$ which is homotopic to a loop γ_i in the simple lamination. If l and m provide the same orientation for γ_i , then $\mathcal{I}_{\mathcal{D}}(l, m)$ is defined as $k/2$ times the minimal number of intersections between l and m . If l and m provide different orientations for γ , then $\mathcal{I}_{\mathcal{D}}(l, m)$ is minus this quantity.
4. Let l_1 and l_2 be laminations on $S_{\mathcal{D}}$ such that no curve from l_1 intersects a curve from l_2 . Then $\mathcal{I}_{\mathcal{D}}(l_1 + l_2, m) = \mathcal{I}_{\mathcal{D}}(l_1, m) + \mathcal{I}_{\mathcal{D}}(l_2, m)$.

In the classical setting of \mathcal{A} - and \mathcal{X} -laminations, the intersection pairing is the tropicalization of the multiplicative canonical pairing. We will now prove the analogous result for \mathcal{D} -laminations.

Proposition 6.16. Let l be an intersecting curve of weight 1 on $S_{\mathcal{D}}$. If γ_i is a component of γ that meets a curve of l , assume that the orientation that l provides for γ_i agrees with the orientation of S . Then we have $\mathcal{I}_{\mathcal{D}}(l, m) = (\mathbb{I}_{\mathcal{D}}(l))^t(m)$ for any $m \in \mathcal{D}_L(S, \mathbb{Z})$.

Proof. Let $m \in \mathcal{D}_L(S, \mathbb{Z})$ be a \mathcal{D} -lamination with coordinates b_j and x_j for $j \in J$, and let e^{Cm} denote the point of $\mathcal{D}^+(S)$ with coordinates

$$B_j = e^{Cb_j}$$

and

$$X_j = e^{Cx_j}$$

for $j \in J$. This point $e^{Cm} \in \mathcal{D}^+(S)$ determines hyperbolic structures on S and S° , so we can view the universal covers \tilde{S} and \tilde{S}° as subsets of the hyperbolic plane.

Cut the intersecting curve l along the image of ∂S and then, using the natural map $S^\circ \rightarrow S$, draw all of the resulting curves on S . These curves form a cycle on S which we can lift to a path in the universal cover \tilde{S} connecting points on $\partial \mathbb{H}$. Choose an ideal triangulation T of S , and lift this to a triangulation \tilde{T} of \tilde{S} . Let P be a triangulated ideal polygon, formed from the triangles of \tilde{T} , that contains all of the triangles that intersect this path. The curves of the path that project to curves from S determine a lamination $u \in \mathcal{X}_L(P)$.

In exactly the same way, we can lift S° to its universal cover \tilde{S}° , and the triangulation \tilde{T} and polygon P determine a triangulation \tilde{T}° and a polygon P° in \tilde{S}° . The curves of the path in P° that project to curves from S° determine a lamination $u^\circ \in \mathcal{X}_L(P^\circ)$.

Now consider the portion of m that intersects S . Lifting these curves to the universal cover, we get a collection of (possibly infinitely many) curves on the polygon P . Let p be any vertex of P . If there are infinitely many curves connecting the sides that meet at p , then we can choose one such curve α_p and delete all of the curves between α_p and the point p . By doing this for each vertex, we remove all but finitely many curves and get a lamination $v \in \mathcal{A}_L(P)$.

In the same way, we can consider the portion of m that intersects S° . Lifting these curves to the universal cover, we get a collection of (possibly infinitely many) curves on the polygon P° . The distinguished curves α_p that we chose near the vertices of P determine corresponding curves α_p° near the vertices of P° . For each p , delete all of the curves on P° between α_p° and p . In this way, we remove all but finitely many curves and get a lamination $v^\circ \in \mathcal{A}_L(P^\circ)$.

For any edge i of the triangulation of P , write a_i for the corresponding coordinate of the lamination v . For any edge i of the triangulation of P° , write a_i° for the corresponding coordinate of the lamination v° . Consider the point e^{Cv} of $\mathcal{A}^+(P)$ with coordinates $A_i = e^{Ca_i}$ and the point e^{Cv° of $\mathcal{A}^+(P^\circ)$ with coordinates $A_i^\circ = e^{Ca_i^\circ}$. Since the cross ratios of the numbers A_i are simply the coordinates X_j introduced above, we know that the point $e^{Cv} \in \mathcal{A}^+(P)$ determines the same hyperbolic structure on P that we get from the point e^{Cm} . Similarly, the point $e^{Cv^\circ} \in \mathcal{A}^+(P^\circ)$ determines the same hyperbolic structure on P° that we get from e^{Cm} . It follows that

$$\mathbb{I}_{\mathcal{D}}(l)(e^{Cm}) = \frac{\mathbb{I}(u^\circ, e^{Cv^\circ})}{\mathbb{I}(u, e^{Cv})}.$$

Using the definition of tropicalization and the result from the classical theory of the \mathcal{A} - and

\mathcal{X} -spaces, we see that

$$\begin{aligned} (\mathbb{I}_{\mathcal{D}}(l))^t(m) &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}_{\mathcal{D}}(l)(e^{Cm})}{C} \\ &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}(u^\circ, e^{Cv^\circ})}{C} - \frac{\log \mathbb{I}(u, e^{Cv})}{C} \\ &= \mathcal{I}_{\mathcal{D}}(l, m) \end{aligned}$$

as desired. \square

Proposition 6.17. Let l be a loop of weight k on $S_{\mathcal{D}}$ which is not an intersecting curve. Then we have $\mathcal{I}_{\mathcal{D}}(l, m) = (\mathbb{I}_{\mathcal{D}}(l))^t(m)$ for any $m \in \mathcal{D}_L(S, \mathbb{Z})$.

Proof. Let $m \in \mathcal{D}_L(S, \mathbb{Z})$ be a \mathcal{D} -lamination with coordinates b_j and x_j ($j \in J$), and let e^{Cm} denote the point of $\mathcal{D}^+(S)$ with coordinates $B_j = e^{Cb_j}$ and $X_j = e^{Cx_j}$.

If l lies entirely on S° and is not homotopic to a curve in the simple lamination, then we define u to be the point of $\mathcal{A}_L(S^\circ)$ obtained by cutting $S_{\mathcal{D}}$. On the other hand, suppose l is homotopic to a loop γ_i in the simple lamination and l and m provide the same orientation for this loop. If the orientation of γ_i agrees with the orientation of S° , then we define u to be the point of $\mathcal{A}_L(S)$ obtained by shifting l onto the surface S and cutting $S_{\mathcal{D}}$. If the orientation agrees with that of S , then we define u to be the point of $\mathcal{A}_L(S^\circ)$ obtained by shifting l onto S° and cutting.

Now let v be the point of $\mathcal{X}_L(S)$ or $\mathcal{X}_L(S^\circ)$ that we get by drawing the lamination m on $S_{\mathcal{D}}$ and cutting. Let e^{Cv} denote the point of $\mathcal{X}^+(S)$ or $\mathcal{X}^+(S^\circ)$ whose coordinates are obtained by scaling the coordinates of v by C and exponentiating. By construction, we have

$$\mathbb{I}_{\mathcal{D}}(l, e^{Cm}) = \mathbb{I}(u, e^{Cv}).$$

It follows that

$$\begin{aligned} (\mathbb{I}_{\mathcal{D}}(l))^t(m) &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}_{\mathcal{D}}(l, e^{Cm})}{C} \\ &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}(u, e^{Cv})}{C} \\ &= (\mathbb{I}(u))^t(v). \end{aligned}$$

Applying the result from the classical theory of the \mathcal{A} - and \mathcal{X} -spaces, we see that the last expression is $k/2$ times the minimal number of intersections between l and m .

If l lies entirely on S and is not homotopic to a curve in the simple lamination, then we define u to be the point of $\mathcal{A}_L(S)$ obtained by cutting $S_{\mathcal{D}}$. On the other hand, suppose l is homotopic to a loop γ_i in the simple lamination and l and m provide different orientations for this loop. If the orientation that l provides for γ_i agrees with the orientation of S , then we define u to be the point of $\mathcal{A}_L(S)$ obtained by shifting l onto the surface S and cutting. If the orientation agrees with that of S° , then we define u to be the point of $\mathcal{A}_L(S^\circ)$ obtained by shifting l onto S° and cutting.

Let v and e^{Cv} be as above. Then by construction, we have

$$\mathbb{I}_{\mathcal{D}}(l, e^{Cm}) = \mathbb{I}(u, e^{Cv})^{-1}$$

so that

$$\begin{aligned} (\mathbb{I}_{\mathcal{D}}(l))^t(m) &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}_{\mathcal{D}}(l, e^{Cm})}{C} \\ &= \lim_{C \rightarrow \infty} \frac{\log \mathbb{I}(u, e^{Cv})^{-1}}{C} \\ &= -(\mathbb{I}(u))^t(v). \end{aligned}$$

Applying the result from the classical theory once again, we see that this last expression is minus $k/2$ times the minimal number of intersections between l and m . \square

Proposition 6.18. Let $l, m \in \mathcal{D}_L(S, \mathbb{Z})$. If γ_i is a component of γ that meets a curve of l , assume that the orientation that l provides for γ_i agrees with the orientation of S . Then $\mathcal{I}_{\mathcal{D}}(l, m) = (\mathbb{I}_{\mathcal{D}}(l))^t(m)$.

Proof. Let $l \in \mathcal{D}_L(S, \mathbb{Z})$. Then we can write $l = l_1 + \cdots + l_d$ where l_1, \dots, l_d are mutually nonintersecting and nonhomotopic closed curves. We then have

$$\begin{aligned} \mathcal{I}_{\mathcal{D}}(l, m) &= \mathcal{I}_{\mathcal{D}}(l_1, m) + \cdots + \mathcal{I}_{\mathcal{D}}(l_d, m) \\ &= (\mathbb{I}_{\mathcal{D}}(l_1))^t(m) + \cdots + (\mathbb{I}_{\mathcal{D}}(l_d))^t(m) \\ &= (\mathbb{I}_{\mathcal{D}}(l_1) \cdots \mathbb{I}_{\mathcal{D}}(l_d))^t(m) \\ &= (\mathbb{I}_{\mathcal{D}}(l))^t(m) \end{aligned}$$

by Propositions 6.16 and 6.17 and the multiplicativity and additivity of the maps $\mathbb{I}_{\mathcal{D}}$ and $\mathcal{I}_{\mathcal{D}}$. \square

We can naturally extend $\mathcal{I}_{\mathcal{D}}$ to a pairing $\mathcal{D}_L(S, \mathbb{Z}) \times \mathcal{D}_L(S, \mathbb{Z}) \rightarrow \mathbb{Q}$ using the tropicalization of the rational function κ_i defined above.

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